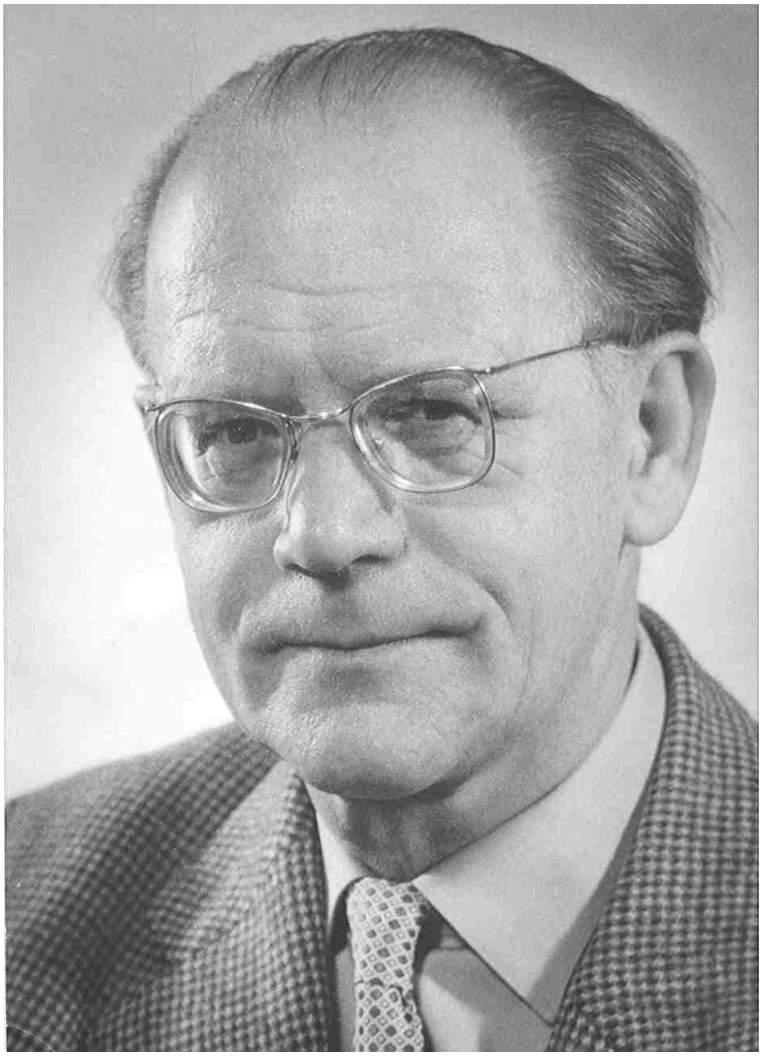

Essays on Georg Rasch and his contributions to statistics

PhD thesis

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Summary

This PhD thesis consists of six self-contained articles. All of them are connected to Georg Rasch and his contributions to statistics.

Chapter 1 is a biography of Georg Rasch covering the years 1901 to the late 1940s. Special attention is paid Rasch's road from being a mathematician to becoming a statistician. His life in the late 1940s is also described at some length. He was a statistical consultant, but he also lectured on mathematics and statistics at the University of Copenhagen. This way, and especially by influencing his young assistants, Rasch was a prominent figure in the progress of statistics in Denmark.

Chapter 2 rediscovers Rasch's work on Growth. We shall see the theory that Rasch developed and explore the events that inspired him to analyse growth. The Growth Model had it's origin in an empirical analysis of calves, but afterwards Rasch used it to model the growth of other animals. By time, Rasch became intrigued by developing a theoretical frame for the Growth Model.

Chapter 2 also includes an empirical example of using the Growth Model on recent data.

Chapter 3 is an account of how Rasch developed the Rasch models. Rasch was triggered by an empirical analysis and in the course of some ten years he gained the insight that the idea underlying this particular analysis was a special case of something much more general, namely specific objectivity.

Chapter 3 also supplies background information on Rasch prior to his development of the Rasch model, but the main focus stays at the late 1940s and the 1950s.

Chapter 4 describes Rasch as professor of Statistics. It would be wrong to say that Rasch's professorship was a indisputable success. First of all, Rasch had hoped to be professor at another faculty. Second, some of the scientists at the faculty where he became professor had hoped for somebody else in Rasch's place.

As professor, Rasch developed the course in statistics. This change was very wel-

come to a segment of students and scientists, namely the sociologists. But a larger segment of people, namely the economists, found that the course in statistics had become next to useless.

Towards his colleagues Rasch was at the same time very generous and very domineering. He was good at initiating projects and successful in creating a fruitful working environment.

Chapter 5 departures from Georg Rasch's ideas on specific testing. Specific testing is combined with exact testing and the existing results on specific testing are generalized.

The main idea of exact Rasch testing is the test statistics used to check the fit of the model is a both necessary and sufficient condition for the model in question. This implies that the observed value of the test statistic is extreme if and only if the observation is extreme as distributed according to the model.

The article discloses a class of exponential families for m-dimensional contingency tables for which it is possible to make an exact Rasch test of the goodness of fit of the model. Furthermore, an exact Rasch test of Rasch's multidimensional Rasch model with multidimensional parameters is derived.

Appendix A, *Pharmaceutical Prices in Europe: A Linear Structure*, derives a pharmaceutical price index. An overall structure in the data is identified and modelled by a two sided analysis of variance. The estimates constitute the price index. This approach is fundamentally different from the commonly used Laspeyres and Paasche index formulas.

The connection of this paper to the work of Rasch is stated in the introduction.

Contents

Introduction

This PhD thesis explores Georg Rasch and some of his contributions to statistics. It consists of 6 self-contained articles, structured as 5 chapters and an appendix. The Chapters 1 through 5 are ordered chronological according to the time in Rasch's life they are connected to. Combined they portrays diverse aspects of Rasch's life until the time of his retirement. The introduction establishes how each article fits in.

Rasch is probably best known for his development of the Rasch models, which is a class of statistical models designed for the analysis of questionnaires. Although Rasch published relatively few articles about these models, they are today widely used in educational and psychological statistics. Biographical information about their origin and their creator is therefore believed to be of interest to scientists within these areas. Rasch's influence was, however, not restricted to the Rasch models. Despite the fact that he was not given an official position in academia until he was 60 years old, Rasch was a prominent figure in spreading R.A. Fisher's new ideas on mathematical statistics to Denmark. Furthermore, being a statistical consultant, he influenced many a doctor's thesis in Denmark.

Rasch was an independent thinker who did not care much about the Literature; both when it came to reading it and when it came to writing it. He preferred to think for himself, and obviously found it tiresome to write down his thoughts. These traits makes him interesting to study, because whatever theory he developed is likely be independent of the existing results. Furthermore, some of the theory he developed still remains undiscovered.

It is the belief of the author that it is important to study the life of Rasch and his contributions to statistics. Not just to establish his influence and to document his originality in developing the Rasch models, but also to examine some of the potential important discoveries on statistics that Rasch did not publish and which might otherwise have been forgotten.

Little has been written about Rasch. Therefore, the main sources to this thesis are interviews with people who knew Rasch together with an interview with

Rasch, carried out by David Andrich in the summer of 1979. I would like to thank David Andrich for allowing me to use his interview (the Andrich-Interview for further references). Furthermore, I would like to thank Peter Allerup, David Andrich, Ellen Andersen, Erling Andersen, Michael Weis Bentzon, Hans Brøns, Ulf Christiansen, Christian Groth, Anders Hjort Hald, Arne Jensen, Niels-Erik Jensen, Benny Karpatschof, Niels Kærgård, Poul Milhøj, Axel Mossin, Lis Olsen, Eggert Petersen, Børge Prien, Jon Stene, Agnete Toft, Hans Vejleskov and Anders Ølgaard for telling me about their memories of Rasch.

Yet a source of inestimable importance is Rasch's library, which is the name we shall use for the papers and books Rasch had in his home office. When Rasch was about 70 he moved from his large villa in Holte to his weekend cottage on the Danish island Læsø. On this occasion Rasch's friend, Arne Jensen, moved Rasch's library to the Danish University of Technology. Here, it has been stored, almost untouched, until the spring semester of 2001 when it was moved to the present author's office at Studiestræde 6. Rasch's library has now, with the conclusion of this project, been moved to the Danish University of Education, a place to where Rasch was strongly connected.

Chapter 1, *Georg Rasch prior to the Rasch models*, explores Rasch's background and how he became a statistician. We shall see which kind of scientist Rasch was by describing his background. Chapter 1 covers the period from 1901 until the late 1940s.

Rasch was born in 1901. In his youth he studied mathematics at the University of Copenhagen, Faculty of Mathematics and Natural sciences. His hopes were to make himself a future in academia as a mathematician. These hopes showed themselves to be fruitless, as the only position available was given to somebody else. To maintain a living, Rasch began to take on minor jobs, helping various scientists with the problems they were facing when analysing data. This way, Rasch slowly turned to statistics, and because he had the backing of influential scientists, he was awarded a Rockefeller scholarship for a years study with the famous statistician, R.A. Fisher.

When Rasch returned from his stay with Fisher he was chiefly occupied as a statistical consultant; in the beginning especially at the Hygienic Institute and the State Serum Institute. Rasch was also affiliated the University of Copenhagen, where he lectured on mathematics and statistics. This way, and by influencing his young assistants at the State Serum Institute, Rasch was a prominent figure in the progress of statistics in Denmark.

The years when Rasch was abroad studying were 1935 and 1936. From his return and until the early 1950s he was mostly consulted by scientist from the medical world; especially by doctors who planned to write a doctoral thesis. Often, Rasch's statistical contribution was published as a (separate) part of the client's thesis, and this way Rasch published some 50 experimental papers.

Rasch had a habit about wanting to develop his own theory instead of searching the literature for a solution, and he hereby got a lot of interesting ideas. Working as a consultant, and being much more interested in thinking than in writing, Rasch seldom got time for finishing these ideas for publication. Actually, there is no telling how many good ideas have gone lost over the years. In this PhD thesis two of such 'lost' ideas have been retrieved. They are respectively about specific testing and a longitudinal model for growth. Specific testing pertains to the late 1960s, and will be further described when introducing Chapter 5, *Exact Rasch Testing*. The model for growth was developed in the 1940s and early 1950s, that is, when doctors and the like consulted Rasch.

Chapter 2, *Georg Rasch's Growth Model*, explores Rasch's work on growth. Looked upon as an independent article, the object is to analyse Rasch's scientific progress. As a chapter of the present thesis, however, the article exemplifies how Rasch developed theory. Empirical problems would prompt him to make progress. And afterwards, maybe even in the course of several years, Rasch would develop a statistical framework for his ideas.

The event that set Rasch going on developing the Growth Model was that he in 1940 acted as critic at Steensberg's defence of his doctoral thesis. Steensberg had carried out a large-scale growth experiment. Confronted with Steensberg's empirical problem, not to mention the flaws and errors in Steensberg's analysis, Rasch discovered the Growth Model as an alternative way of analysing the data.

During the 1940s Rasch used the Growth Model to analyse other data sets on growth, and these analyses forced him to make progress. In 1951, when the International Statistical Institute held their meeting in India, Rasch was invited to give a series of lectures in Calcutta on his work on growth. On this occasion he wrote a set of lecture notes, consisting of some 200 hand written pages. Half of them were raw data, numerical computations and graphs, while the other half were theory – at times quite coherent, but at other times having the appearance of a very rough draft. Knowing how little patience Rasch had with writing things down, it is very

likely that his Growth Model gradually would have disappeared had he not had the Calcutta lectures as a motive for writing down his thoughts.

When Rasch returned from Calcutta he polished the statistical theory for the Growth Model. For the next couples of years he worked on constructing a stochastic process that was a solution to the difference equation underlying the Growth Model. His work culminated in 1954 when he gave a seminar at the University of Copenhagen on a solution to a multidimensional stochastic difference equation.

In the mid 1950s Rasch put his Growth Model on the shelf. At this point in time he was deeply involved in the analysis of questionnaires and attainment tests, that is, in developing the Rasch models. He therefore had little time, nor use, for a growth model. Rasch did not really use the Growth Model again until the late 1960s. At this point in time his main interest in science was the concept of specific objectivity, which originated in the Rasch models. It seems that Rasch found it very interesting that the Growth Model, in some sense, was a special case of specific objectivity.

Chapter 2 pays special attention to how Rasch developed the Growth Model, but it also contains a new empirical example. The data describes the growth in capital per worker in various OECD countries in the period 1951 to 1990. The main issue is to illustrate the Growth Model empirically; not on analysing the specific data. The parameters of the Growth Model have also been estimated by using the theory of stochastic processes that are available today.

Rasch is probably best known for the class of statistical models that bears his name, namely the Rasch models. Chapter 3, *The life of Georg Rasch as a Mathematician and as a Statistician*, explores Rasch's development of these models. The paper, which is written jointly with Erling B. Andersen, has been published in Essays on Item Response Theory, edited by Anne Boomsma, Marijtje A. J. van Duijn, and Tom A.B.Snijders. Looked upon as a chapter of this thesis, Chapter 3 is yet an example of how Rasch developed theory; he would be triggered by an empirical problem, and his solution would by time evolve into fundamentally new theory. Chapter 3 pays special attention to the period 1950 to 1960. It was in the span of those 10 years relatively late in Rasch's life that he developed most of the statistical theory for the Rasch models.

Rasch's development of the Rasch models began in 1951 when the Danish Ministry of Social Affairs consulted Rasch with view to find out whether extra education given to kids suffering from dyslexia had an enduring effect. The techniques Rasch developed to analyse these kids' reading progress underlies the Rasch models.

In the early 1950s Rasch was increasingly consulted by the Military Psychology Group, where he was permitted, even encouraged, to do research. When Rasch got the notion of using the techniques he had developed when analysing the kids suffering from dyslexia to analyse the intelligence of recruits, he was given free rein. Furthermore, a small group of people was placed at Rasch's disposal with a view to construct a new intelligence test.

By the establishment of the Danish Institute of Educational Research in 1955 Rasch concentrated much of his consulting here. The director, Erik Thomsen, was of the opinion that Rasch's ideas on analysing attainment tests ought to be published, and he therefore paid Rasch to write a book on his discoveries. The resulting book, Rasch (1960), together with Rasch's presentation of the Rasch models at the Fourth Berkeley Symposium on Mathematical Statistics and Theory of Probability (Rasch, 1961), made Rasch famous.

Much more could be said about the Rasch models than is the case in this thesis, and much more has already been said. The Rasch models and specific objectivity are those of Rasch's contributions to statistics that he developed and published most fully. And it was without doubt those he cared most about. Furthermore, several prominent scientists have further developed the theory for Rasch models (see for instance Fischer and Molenaar (1995) and Boomsma et al. (2001).). But, still, if Rasch hadn't had such excellent opportunities for further developing the Rasch models at the Military Psychology Group and the Danish Institute of Educational Research, in all likelihood, the Rasch models would have remained as unknown as the Growth Model.

Chapter 4, *Georg Rasch. Professor of Statistics at the Faculty of Social Sciences, 1962 to 1971*, describes Rasch as professor at the University of Copenhagen. It was the paper I least wanted to write, the point being that it was rumoured that Rasch was not well liked as professor. As the paper progressed it turned out that the circumstances were subtler than first anticipated.

Rasch was 61 years old when he was appointed professor. At this point in time he had just become internationally recognized for the Rasch models (though the models were not yet referred to by that name). However, Rasch was not quite happy with his chair in statistics, since it was placed at the Faculty of Social Science, and not at the Faculty of Mathematics and Natural Sciences. He was of the opinion that influential mathematicians thwarted his opportunities of ever becoming professor at the Faculty of Mathematics and Natural Sciences. They had done so back in the

1930s when Rasch tried for a future as a mathematician. And now, about 1960, they thwarted his opportunities of applying for the newly established chair in statistics at the faculty of Mathematics and Natural Sciences. So, when Rasch applied for the chair in statistics at the Faculty of Social Sciences, he was in an embittered state of mind. To make things even worse, influential scientists at the Faculty of Social Sciences had another scientist as their first choice for the chair, namely E. Lykke Jensen. So, the beginning was not that good.

When in 1962 Rasch was appointed professor, all professors reigned as kings within their domains. Rasch had strong opinions and few hesitations about letting his opinions dictate his curriculum. This was also the case for the other professors, but the problem was that Rasch thought that the commonly used statistical methods were unscientifically used. He therefore removed these methods from the course in statistics and put in its place statistical models of his own invention, namely the Rasch models.

At the Faculty of Social Sciences statistics was an auxiliary subject for economics and sociology. The sociologists approved of the changes Rasch made, but the economists most certainly did not. The impact was that Rasch and the economists had little to do with each other, and little desire for this state of affair to change. By time, such circumstances have evolved into the rumour that Rasch was not well liked. Chapter 4 explores these circumstances. The working climate when Rasch was professor is also briefly described. Rasch was domineering, but he was also caring towards his associates. He was good at initiating projects and while he was professor he created a fruitful working environment for sciences.

Chapter 5, *Exact Rasch Testing*, belongs to the late 1960s and the 1970s. At this point in time Rasch was on the verge of retiring as professor in 1971. However, he still developed fundamentally new ideas; in this case specific testing. In contrast to the preceding chapters, Exact Rasch Testing aims at a further development of Rasch's idea, while no information is given about how Rasch discovered specific testing. With a view to place Exact Rasch Testing in the context of the other Chapters a summary of the circumstances will be given here:

The Ministry of Transport wanted to know if speed limits had any reducing effect on the number of traffic accidents, and they therefore consulted Rasch.[1] As a starting point he tried on a multiplicative Poisson model, and for some of the questions

[1]The background information is based on the Andrich-Interview and Rasch (1968).

examined it worked fine, but for others it was inadequate. More specifically Rasch was examining whether the number of traffic accidents at a given day of the week, X_{ji}, at week number j, $j = 1, ..., 52$, of the year i, $i = 1961, 1962$, could be described by a multiplicative Poisson model with mean $\theta_j \sigma_i$, where the parameter θ_j pertained to the specific year, and σ_i pertained to the specific week of the year.

One of Rasch's characteristics in empirical modelling was that he was very thorough in checking to which degree the data conformed to the suggested model. In the specific case Rasch checked the multiplicative Poisson model by examining whether the conditional distribution of the X_{ji}'s given the marginals, e.g. $x_{j.} = \sum_i x_{ji}$, were binomials.[2] He did this by plotting $(x_{ij}, x_{j.})_{j=1,....,52}$ for each year and for all the days of the week. What the plots revealed was that the points scattered nicely around a straight line with slope corresponding to the mean in the binomial, namely $x_{j.} \cdot \frac{\sigma_i}{\sigma_.}$. However, the variance was in general larger than $x_{j.} \cdot \frac{\sigma_i}{\sigma_.}(1 - \frac{\sigma_i}{\sigma_.})$.

Rasch therefore formulated mathematically the requests that the conditional distribution had mean as in the binomial, but a larger variance. And from these requests he derived that the only distribution that fulfilled these requests was a negative binomial. He therefore used this model to analyse the data.

More importantly than finding a usable model, Rasch had found out that the conditional mean and variance characterized the negative binomial completely. That is, knowing that the distribution in question had this conditional mean and that conditional variance, the distribution was bound to be a negative binomial. Other scientists had also found conditions that characterized various distributions completely, but the important thing was that Rasch immediately saw the implication in light of checking the fit of a statistical model. Namely that if it was possible to check the fit of the model by using a probabilistic statement that was not only a consequence of the model, but which in fact implied the model, then this check of fit of the model was more reliable than other types of tests. Rasch said for such cases that it was possible to make a specific test of the model in question.

The first paper Rasch wrote about specific testing was the analysis of traffic accidents (Rasch, 1968). Other known sources on specific testing are Rasch (1971) and Rasch (1974) together with a not dated report carrying the title *On Objectivity and Specificity of the Probabilistic Basis for Testing* (Rasch, 2001). What these article demonstrate is that Rasch found specific testing very interesting, and that he further developed it.

[2] $(X_{ji} \mid X_{j.} = x_{j.}) \sim bin(x_{j.}, \frac{\sigma_i}{\sigma_.})$

Specific testing is better preserved than for instance the Growth Model. However, Rasch never published about it in an international journal. Adam Gottschau, who in his master's thesis wrote about Rasch's work on psychometrics, has further developed specific testing. More specifically, Gottschau has shown that a class of exponential families for two dimensional contingency tables are completely characterized by their conditional distributions given the sufficient statistics.

Chapter 5 is a further development of Rasch's ideas on specific testing into Exact Rasch Testing. One of the contributions is that Gottschau's result has been generalized to multidimensional contingency tables. Furthermore, specific testing is combined with exact testing. Finally, Chapter 5 includes an exact Rasch test of Rasch's multidimensional Rasch model with multidimensional parameters.

There is no doubt that Rasch was aware of the exact test approach. And, there exist a couple of reports where he suggests exact tests, for instance his text book (Christiansen and Stene (1968)). In general, however, Rasch was not too fond of tests, the point being that an overall test only gives information as to whether the model fits the data or not, while the question of *why* the model does not fit is left unanswered (Andersen, 1995). Therefore, there is no telling if Rasch would have approved of exact Rasch testing.

The last article of this thesis, *Pharmaceutical Prices in Europe: A Linear Structure*, is placed in an appendix, the point being that its relation to the work of Rasch is less obvious than is the case for the five other papers. The article is written jointly with Tim Bedsted, Kasper P. Jørgensen and Hans Keiding. Its main contribution is to derive a pharmaceutical price index. The paper identifies an overall structure in the data. Modelling this structure by a two-sided analysis of variance, at the same time checking for bias due to various exogenous variables, a price index is estimated. This approach is fundamentally different from the commonly used Laspeyres and Paasche index formulas.

A connection of Appendix A to the work of Rasch is that Rasch was a spokesman for interdisciplinary research. As is exemplified in the two articles, *Georg Rasch's Growth Model* and *The life of Georg Rasch as a Mathematician and as a Statistician*, Rasch often worked with scientists from other branches of sciences on empirical projects – on many occasions developing something new in the process. It therefore seemed appropriate for the present author to try out working with non-statistical scientists on an empirical project.

The empirical problem of Appendix A was that few drugs were marketed in all

countries, implying that the commonly used index formulas should be used with great precaution. Trying instead a different approach, namely statistical modelling, it turned out that a two sided analysis of variance fitted the data quite nicely. This seems to be an example of that Rasch had a point in being an advocate for interdisciplinary research. In this case a combination of health economics and a simple statistical model lead to something new.

Another connection between Appendix A and the work of Rasch is that the empirical analysis underlying the article follows Rasch's methodology when doing empirical work. Rasch was said to be very thorough in the initial steps of an analysis. He would always make a lot of plots. And first when these plots revealed the structure of the data would he proceed. This approach was followed and the data were plotted in all thinkable ways. Actually, 153 plots are our argumentation for parameterizing the mean as we have. Furthermore, the subsequent analysis of whether exogenous variables gave rise to derivations from the suggested model was very thorough. We have, though, differed from Rasch at one important point, namely in modelling the errors by a normal distribution. Rasch used the normal distribution himself; at least in the 1940s and 1950s (see for instance Chapter 2). In the 1960s, however, he developed a very censorious attitude towards it (see for instance Chapter 4).

The last but also most obvious connection of Appendix A to the work of Rasch is that Rasch, of course, has a publication in which he parameterised the mean as is done in a two sided analysis of variance, namely Rasch (1950). And, in this paper Rasch used plots to motivate the model; just as is done in the article *Pharmaceutical Prices in Europe: A Linear Structure*.

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I would like use this opportunity to thank a number of colleagues and friends. First of all, I would like to thank my supervisor, Anders Milhøj, for his help and encouragement and for always reminding me that three years is not an eternity when it comes to writing a thesis. I would also like to thank Erling B. Andersen for encouraging me to begin this PhD project, and especially for involving me in the article, *The life of Georg Rasch as a Mathematician and as a Statistician*. Special thanks to Peter Allerup, David Andrich, Martin Jacobsen, Svend Kreiner, Paula Madsen, Bo Markussen and Helle Sørensen for providing inspiration and valuable contributions. Furthermore, I would like to thank my co-authors, Erling B. Andersen, Tim Bedsted, Kasper P. Jørgensen and Hans Keiding for a productive cooperation. Last, but not least I would like to thank members of my family, Aage T. Andersen,

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1 References

Andersen, Erling, B. (1995). What Georg Rasch Would Have Thought about this Book. In Gerhard H, Fischer, Ivo W Molenaar (Eds.), *Rasch Models*. Springer-Verlag New-York, 383−390).

Anne Boomsma, Marijtje A. J. van Duijn, Tom A.B.Snijders (2001). *Essays on Item Response Theory.* Lecture Notes in Statistics. Springer-Verlag New York, Inc.

Ulf Christiansen and Jon Stene (1968). *G. Rasch's lærebog i statistik*, bind 1. København.

Gerhard H. Fischer, Ivo W. Molenaar (1995). *Rasch Models*. Springer-Verlag New-York.

Gottschau, Adam. (1983), *En Analyse af et Objektivitetsbegreb. Med Udgangspunkt i Georg Rasch's Psykometriske Arbejder*. Danmarks Pædagogiske Institut. Copenhagen.

Rasch, Georg (1950). Statistik og forsøg. Orientering i et materiale af markforsøg over kvælstofgødninger. *Tidsskrift for Planteavl*, 53, pp. 369-401.

Rasch, G, Larsen, C.A. and Ellehammer, M.. (1952). *Statistiske undersøgelser over nuværende og tidligere læseklasseelever i København.* Rapport til kommissionen angående omorganisation af forsorgen for talelidende. Arbejds- og Social-ministeriet. Typewritten. Copenhagen.

Rasch, G. (1968). *En reanalyse af danske og svenske forsøg over virkningen af hastighedsbegrænsninger på trafikulykker.* Typewritten report. Duplicated. Copenhagen.

Rasch, G. (1971). *Proof that the necessary condition for the validity of the multiplicative dichotomic model is also sufficient.* Duplicated. Typewritten. Universitetets Statistiske Institut. Copenhagen.

Rasch, G. (1974). On Specific Distributions for Testing of Hypotheses. *Memoirs* no.1

1974. Proceedings of conference on foundational questions in statistical inference. Ed. O. Barndorff-Nielsen, P. Blæsild, G. Schou, University of Aarhus, Department of theoretical Statistics, pp. 101-110.

Rasch, G. (2001). On Objectivity and Specificity of the Probabilistic Basis for Testing. In *Rasch Lectures. In honor of Georg Rasch's 100 years birthday on the 21th of September, 2001.* Eds. L. Olsen and S. Kreiner. Copenhagen Business School.

Georg Rasch prior to the Rasch models

Lina Wøhlk Olsen[1]

Chapter 1

ABSTRACT This article explores Georg Rasch's life prior to his development of the Rasch models. It will be described how Rasch in his youth studied mathematics, and how he at the quite young age of 30 defended his doctoral thesis. As it was, there was no available positions in mathematics for Rasch and he turned towards statistics. He was granted a scholarship to study statistics with R.A. Fisher; a circumstance that influenced the progress of statistics in Denmark. Rasch's main occupation before he published the Rasch models will also be described at some length. He worked as a statistical consultant, and through his empirical work he developed a habit of developing whatever statistical tools he needed for the analysis.

1 Introduction

Georg Rasch was born in 1901 and he died at the age of 79 in 1980. As a scientist he became known for developing the statistical models often referred to as *the Rasch models*. This article portrays Rasch before he began to develop the Rasch models. The objective is to illustrate what kind of scientist he was by describing his background.

In Sections 2 through 6 we will see how Rasch became a statistician in the first place, because Rasch originally had other plans for his future, namely to be a professor of mathematics. Though this plan was not practicable, Rasch continued to define himself as a mathematician throughout his life.

In Section 6 Rasch's training in statistics will be described at some length, the point being that Rasch spent a sabbatical year with R.A. Fisher, undoubtedly the most famous and influential statistician who ever lived. Rasch's encounter with Fisher formed Rasch's point of view on statistics, and, as will be described in Section 10, a side effect was that Rasch, at some level, influenced the progress of statistics in Denmark.

Sections 2 through 6 describe Rasch's life chronologically while the sections that follow are ordered according to topics. The objective is to give a broad picture of Rasch's life in the late 1940s and 1950s. All topics cannot be covered, but it is the

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FIGURE 1. Georg Rasch. Permission was kindly grated by Rasch's sister, Agnete Toft, to reprint this photograph.

belief that those explored give a fair description of Rasch.

Section 7 describes Rasch's main source of income, namely to work as a statistical consultant. In this empirical work Rasch showed himself to be an independent thinker with a habit of developing his own statistical methods rather than searching the Literature for a solution. In Section 8, Rasch's family life will be explored, the point being that Rasch was a social person, and that this trait was important for his work. Following on from this we see that Rasch indeed was involved in sciences. Finally, Section 10 describes Rasch's role as a communicator of Fisher's ideas on statistics.

The main source of this account is a series of interviews with Georg Rasch carried out by David Andrich in the summer of 1979 on the Danish Island of Lœsø, where Rasch lived in his retirement. We refer to this series of interviews as *the Andrich-Interview*. The Andrich-Interview has been compared with interviews carried out by the present author with people who knew Rasch personally. I would like to thank David Andrich for allowing me to use the Andrich-Interview. Furthermore, I would like to thank Peter Allerup, David Andrich, Ellen Andersen, Michael Weis Bentzon, Christian Groth, Arne Jensen, Benny Karpatschof, Axel Mossin, Børge Prien and Agnete Toft for telling me about their memories of Rasch. A brief description of these people's connection to Rasch is given in Appendix A.

As to the written sources, various books and papers on academia in Denmark have been consulted, but the most interesting written sources are personal letters and scientific papers written by Rasch. These sources were found in Rasch's personal library, which is the name we shall use for the papers, books, numerical computations and sundry notes that Rasch had in his home office, and which the present author was permitted to go through.

2 Childhood

Georg William Rasch was born on 21 September 1901.[2] His parents, Johanne *nee* Duusgaard and Vilhelm Rasch were married in 1896. They had five children together: Oskar, Ejnar, Ellen, Georg and Astrid. Rasch's mother was ill during most of his childhood and according to Rasch, she did not influence him much. She died in 1920. Rasch's father, Vilhelm Rasch, was, on the other hand, a strong figure in

[2]Section 2 is based on the Andrich-Interview, DBL (p. 643) and Int. Toft (10.10.2001).

Rasch's life. He was, as Rasch puts it in the Andrich-Interview, "*one of the most hard-boiled evangelists that I have ever known.*" Vilhelm Rasch started his working life as a sailor in 1882. In 1886 he passed his mate's examination and was thereafter engaged in overseas trade. In 1893 he began to teach sailors at the nautical school in Ålborg. In 1897 he entered the Danish Seamen's Mission[3] and after some years he managed to establish a school[4] for sailors in Svendborg with himself as the first headmaster. The family thus moved from Odense, where Rasch was born, to Svendborg in 1906.

Rasch's interest in mathematics was awakened in the public school of Svendborg when he was some 12 or 13 years old. Rasch had the good fortune of getting a new math teacher, Mr. Lehn, whose instructive teaching had an inspiring influence on Rasch. Rasch then began to read some of his father's books in mathematics. These books were about navigation, including plane and spherical geometry. The algebraic manipulations in the books fascinated Rasch immensely.

Fortunately for Rasch, one day his father and Mr. Lehn were on the same train. During this journey Vilhelm Rasch told Lehn that he planned to give his son a higher education by letting him attend secondary school in Svendborg, where the family lived. According to Rasch's statements in the Andrich-Interview, Lehn, who had recognized Rasch's abilities in mathematics, said:

> *No Mr Rasch, do not do that. Your son is a gifted mathematician, you must take care that he gets in a secondary school where he learns some proper mathematics.*

Vilhelm Rasch then decided that despite the considerable extra expenses, he would send Rasch away from home to attend the high school in Odense. Here Rasch lived for the next three years while finishing school.

3 Student

After graduating from high school in 1919 Rasch began to study mathematics at the University of Copenhagen.[5] At that time there were only two professors of mathematics at the University of Copenhagen, Niels Nielsen and Johannes Hjelmslev.

[3] In Danish: Sømandsmisionen

[4] In Danish: højskole.

[5] Section 3 is based on the Andrich-Interview, Einar Andersen (1983), Bang (1988), Bohr (1931), DBL (p. 643), Nørlund (1926), Ramskov (1995).

Both were highly respected, but Rasch soon came to work for Nielsen. Nielsen did research in the theory of functions, algebra and number series and in his youth Nielsen was very productive. Rasch's work for Nielsen mostly concerned computations within Nielsen's fields of research, and according to Rasch, it was this work that sparked his interest in number series and algebra. Most remarkable, however, is that Rasch wrote an article together with Nielsen while still an undergraduate, namely "*Notes Supplémentaires sur les Equations de Lagrange*" published in 1923 (Nielsen and Rasch, 1923). The article concerns the theory of Lagrange equations, which was of special interest to Rasch. As apparent in the Andrich-Interview, Rasch was quite proud of publishing a paper so young:

> *I sent a copy to my former teacher, Mr. Lehn, who had made me into a mathematician. He was quite jubilant when he got this paper and told me later that he had told his class that he had had a wonderful student, this Georg Rasch who at the quite young age of 21 years published a paper. That was my first highlight in mathematics*

Rasch's work for Nielsen did not continue throughout Rasch's time as a student. At that time, students who had finished the first part of their studies could apply for a scholarship to finance their further studies. Rasch applied and was granted a scholarship at the Regensen, a prestigious hall of residence. Rasch said in the Andrich-Interview:

> *Then I got a grant for my further studies. I became a member of the college called Regensen in Copenhagen where we got free room and board. So I was quite well off for the time until 1925 when I graduated.*
> *When I got that stipend, I did not see any further reason for doing arithmetical work for a living. So I left the work with Professor Niels Nielsen.*

Among his fellow students at the Regensen, Rasch was considered to be a gifted student. Erling Andersen, student of Rasch in the early 1960s and successor to his chair as professor of statistics, writes (Andersen, 1995, p. 386):

> *My father, who was also a mathematician, once told me that Georg Rasch was quite famous already as a graduate student for being able to solve complicated differential equations. One story, which my father claimed was true, went that one of the famous mathematics professors showed up one morning at Georg Rasch's dormitory, while he was shaving, to unravel a riddle connected with a complicated differential equation.*

The mathematics professor mentioned in the quote from Andersen was Harald Bohr, brother to the nuclear physicist, Niels Bohr. During the period 1915 to 1930, Harald Bohr was a professor of mathematics at the Polytechnical College.[6] In the beginning of the 1920s, lectures in basic mathematics were still given jointly to students at the University of Copenhagen and students at the Polytechnical College, so Rasch attended courses given by Harald Bohr.

In 1922 a new chair in mathematics was established at the University of Copenhagen. Creating a new chair was quite a revolution, but Danish mathematicians feared that Niels Erik Nørlund would accept a chair in another country and hereby be lost to Danish mathematics if he was not given a chair in mathematics at the University of Copenhagen. In the Andrich Interview Rasch said:

> *Then Nørlund appeared on the scene. He gave a long series of excellent lectures on a great variety of topics in connection with the theory of analytic functions, functional equations, and all those kinds of things. That interested me immensely.*

Approximately a year after Nørlund's inauguration, which was in 1923, Rasch began to work for him. There is some indications that Nørlund tried to get Rasch a position as an assistant at about this time, however, the application was not fully granted, only some money was made available but not an office (Aarbog, 1924-1925, p. 168-169).

In 1925 Rasch graduated as masters in science.[7] The subject for his dissertation covered some of the areas he had been occupied with while working for Nielsen. Rasch said in the Andrich-Interview:

> *I had got very interested in the number theory. When qualifying in the last examination I had to deliver a paper of some sort.* [...] *I actually delivered a substantial paper, never published, in number theory.*

4 The time as a mathematician

In Denmark in the early 1900s, there were very few possibilities of working as a mathematician at an academic level.[8] Apart from a handful of professorships there were

[6] In Danish, 1930: Polyteknisk Læreanstalt

[7] In Danish: mag.scient

[8] Section 3 is based on the Andrich-Interview, Einar Andersen (1983), Bang (1988), Ramskov (1995).

only some minor positions for assistants. After graduating Rasch had several minor jobs. From 1925 to 1934 Rasch worked for a couple of hours a week at the Geodetic Institute where Nørlund was the director. In the spring of 1925 Rasch became Nørlund's assistant, which also entailed tutorials at the University of Copenhagen (Festskrift, 1930, p. 190). Actually, Rasch continued to give tutorials as Nørlund's assistant until 1940. Apart from the odd tutorial, Nørlund also entrusted Rasch with an important research project in 1925, namely to find out whether a proof of a certain theorem could be found among the papers left behind by J.L.W.V Jensen. Rasch said in the Andrich-Interview:

> *J.L.W.V.Jensen claimed that he had proved Riemann's assertion.*[9] *Now, either in 1922 or early in 1923 he died* [The correct year is 1925]. *Then, of course, every mathematician was very anxious to see if he had left behind him a proof of that theory. Nørlund was, of course, very interested because it was an important question in function theory. Then he got a bit of money to take me on as an assistant with a view to going through the papers left by J.L.W.V. Jensen.*

Dr. phil. J.L.W.V. Jensen was born in 1859 and died 5 March 1925. He was highly respected as a mathematician. His main fields of interest were Riemann's zeta function and the gamma function, and he is especially known for *Jensen's inequality* (Nørlund, 1926, p. 1–7). Apparently, there was no proof of the famous Theorem to be found in the papers left by Jensen. In the Andrich-Interview Rasch said:

> *I did dig something out of it and I got a couple of papers published. One is still unpublished. However not one iota could I find about Riemann's theorem. I worked on the problem for some years and the last paper inspired by it came in the latter part of the 1920s.* [...] *However, it was one of the ways in which I got into research.*

Actually, Rasch published four articles on the gamma function, namely Rasch (1927), Rasch (1928), Hille and Rasch (1928) and Rasch (1931), and these have to be considered a consequence of the research Rasch did while working on Jensen's papers. However, because Rasch was not able to find any proof of Riemann's Theorem, Nørlund finally had to hire the famous mathematician Polya. Polya's conclusion was the same as Rasch's, namely that there was no proof of Riemann's Theorem to

[9] Riemann's assertion has not yet been proved. But it says that all non-trivial zeros, s, of the zeta-function, $\zeta(s)$, have Re$s = 1/2$.

be found (The Andrich-Interview).

In 1928 Rasch married Elna Nielsen. She was always called *Nille*, a charming nickname given to her because of her second name. Nille and Rasch had met each other at Rasch's father's house. After the death of Rasch's mother in 1920, Vilhelm Rasch had left his position as headmaster in Svendborg, and some years later he married Sørine Cathrine Rasmine *nee* Sørensen, always called Kate by the Rasch family. In 1923 their daughter, Agnete, was born. Later, Nille and Rasch would say that they owed their engagement to Agnete. Nille, together with her mother and sister, were frequent guests at Vilhelm and Kate Rasch's house, and Rasch often spent his holiday's at home. Agnete, at this time quite young, tried to hug Rasch and Nille at the same time, in consequence pulling their heads together. This broke the ice, their romance blossomed and they were later were engaged at Vilhelm and Kate Rasch's home (Int. Toft, 10.10.2001).

In 1931 and 1933, respectively, Rasch and Nille's two daughters, Helga and Lotte, were born. In spite of the fact that both Rasch and Nille were brought up in very religious homes, Rasch and Nille were not religious in their adult life, on the contrary one might say. Considering that Rasch's father was a strong figure in the Seamen's Mission this caused a good deal of friction between Rasch and his father (Int. Prien, 08.02.1999), (Int. Toft, 10.10.2001).

On the 12th of July 1930 Rasch defended his doctoral thesis,[10] which in Danish was titled *"Om Matrixregning og dens Anvendelse paa Differens- og Differentialligninger"* (Rasch, 1930). The title translates to *"Matrix Algebra and it's applications to difference and differential equations"*. Rasch wrote about the writing of his thesis in a letter to his friend Carl Einar Hille[11] on the 22 of September 1930:[12]

> *You have asked me for my plans for the future. Yes those that I do have, have evaporated, so now, strictly speaking, I do not have any. The background for this situation is as follows: When it was rumoured that Bohr, under all circumstances, would leave the Polytechnical University I saw the*

[10] The Doctoral thesis carries a much higher level than is required of a PhD thesis.

[11] Carl Einar Hille was a famous mathematician. He was born in USA in 1894, but lived in Sweden until 1920. In 1919 he was awarded the Mittag-Leffler prize. Hille met Rasch when he visited Copenhagen during 1927 (Personal correspondence between Rasch and Hille). At this point in time, Hille was associate professor at Princeton university. The information on Hille is based on an article written by J.J. O'Connor and E.F. Robertson. The article is found at http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Hille.html.

[12] The quote is a translation of the original letter, which is in Danish, see e.g. Appendix B.

FIGURE 2. Rasch standing with Helga, Nille sitting with Lotte. The picture was taken in the garden of Kate and Vilhelm Rasch's house. Permission was kindly grated by Rasch's sister, Agnete Toft, to reprint this photograph.

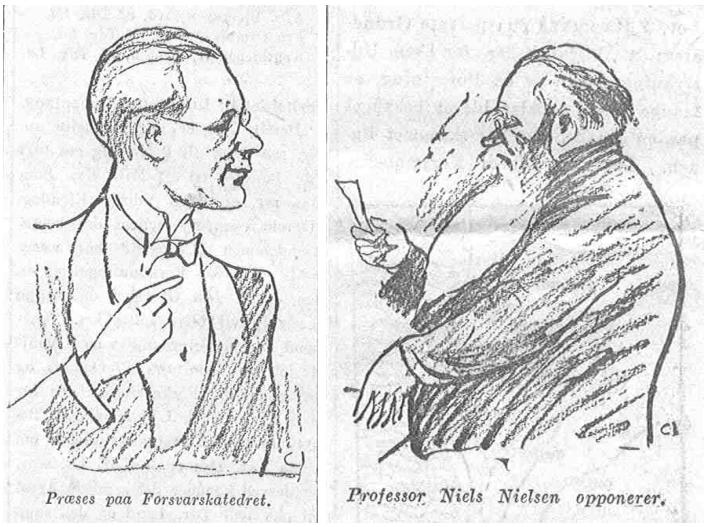


FIGURE 3. Hand drawn pictures from a Danish newspaper. Text to the drawing on the left: Candidate for the doctorate at the lectern of defence. Text to the drawing at the right: Professor Niels Nielsen acts as critic.

chance of a lifetime. It seemed very likely that A.F.Andersen would succeed Bohr, but then, of course, he would have to be replaced as docent at The Royal Veterinary and Agricultural University[13] *and this tenured position was what I was aiming for when I pulled myself together and in seven months wrote and submitted my doctoral thesis. Without much preparation: An idea I had occasionally juggled around with, namely* [...] *I let isotoni be isotoni and plunged directly into the new aspects, whose correspondence made me reappraise my earlier studies on differential equations, and they ended up as Chapter III. However, the process was very hard, because as I was writing up, I realized - over a week's time - to my despair, that at a crucial point, I allowed a straight lined integration path to be deformed into a specific closed curve - which, however, never exists. This happened in the beginning of October. My work crumpled completely, so I had to halt for some 40 days, but by making a vigorous effort I managed to finish my dissertation and submit it to Nørlund before the Christmas vacation, and at the same time, by exerting diplomacy and persuasion, I managed to convince him that it was absolutely necessary for me to defend my thesis before the beginning of the long vacation. I managed to realize the program despite several setbacks. And all of this was staged for the blasted tenured position, which, of course, I did not get after all. While I had been*

[13] In Danish:Landbohøjskolen

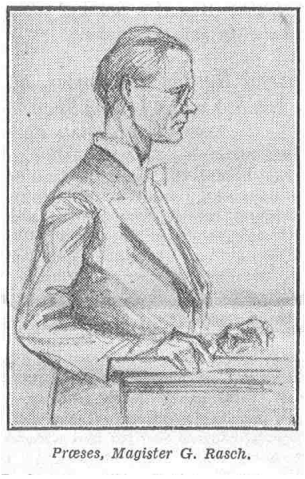


FIGURE 4. Hand drawn pictures from a Danish newspaper. Text to the drawing: Candidate for the doctorate, Master of Science G. Rasch

> *writing, a new star had appeared, namely Børge Jessen, who, according to Bohr, was extraordinarily bright.* [...] *Compared to this stella nova I faded completely and of course I did not have the director of the Polytechnical University as my father in law. However, I wish him well – I only wish, I had any idea what to do next.*

Besides the chairs mentioned in the letter, no permanent position in mathematics became available in Denmark over the next couples of years. It has to be mentioned that Jessen was merely 22 years old when he was appointed professor.

It appears that Rasch was of the opinion that he was cut off from holding a chair in mathematics because he was not a student of Bohr. He said in the Andrich-Interview (but did not like to talk about it):

> *Bohr and I were never very friendly. We were of course on speaking terms, but we never had much to say to each other. He had some other students and these students got professorships in time, but I got nothing.*

Research done by Ramskov supports Rasch's observation that Bohr had a group of young students, and that, in due course, all of these students got a chair in mathematics (Ramskov, 1995, p. 295). Actually, Rasch was of the opinion that when it came to getting a chair, it was by no means an asset to be a student of Nørlund. In the Andrich-Interview he said:

> *I should add that between Bohr and Nørlund there was a conflict. Bohr tried to dominate the Faculty of Science and Nørlund having been called from the University of Lund to the University of Copenhagen, of course felt*

> *that he was a very important person. So they were two antagonist and I as squashed between them* [. . .] *I stuck to Nørlund, and, I have been satisfied with him so far. But he had lost his influence.*

In 1930, after Rasch had defended his thesis, he was almost out of work. He still did tutorials for Nørlund and he still worked for a couple of hours every week at the Geodetic Institute, but that was all. Rasch wrote about his financial situation in a letter to Hille on the 22 of September 1930:[14]

> [...] *continuing to live as I have been doing over the past three years is not something I am very keen on. Frankly speaking I only have 200 Danish crowns per month for living (research assistant to Nørlund and a couple of hours daily at the Geodetic Institute.) On top of that, it seems I have committed the serious blunder of getting married. This state of affairs simply cannot continue, and the whole thing is just about to fall apart. However, I downright refuse to seek an appointment as a teacher or something similar, as I fear that it will put my scientific work at risk.* [...]
>
> *You have naturally long ago guessed that these Lamentations have a rather particular point - for otherwise I would not plague you with them! Honestly speaking! Is there nothing for a white man to do in the USA.* [...] *Isn't there a tenured position somewhere in the USA for a hopeful young Scandinavian mathematician? And if so, would you assist me in obtaining it. Excuse me for being so direct. I realize that I may be asking too much, but this is the only chance of saving my future as a scientist that I can envisage for the present. I am deeply pessimistic about my opportunities in Denmark.*

However, Nørlund managed to get Rasch a scholarship and therefore Rasch did not travel to the States to work as a mathematician. Considering that Rasch's oldest daughter, Helga, was born in 1931 it must have been a great relief for Rasch and Nille to have their financial problems solved. Rasch wrote in another letter to Hille on the 10th of Februrary 1931:[15]

> *Some recognition – besides praise – I am to my great pleasure also receiving in Academia these days, and Nørlund, Niels Nielsen and Hjelmslev have obtained University funding of 1000 Danish crowns per annum for two years (with a possible extension to 4 years), plus a grant from the Carlsberg*

[14]The quote is a translation of the original letter, which is in Danish, see e.g Appendix B.
[15]The quote is a translation of the original letter, which is in Danish, see e.g Appendix B.

> *Foundation of 1200 crowns, so far for one year only, but provided I behave, then renewal is a possibility.* [...]
>
> *This is how matters stand at the moment. My financial circumstances have improved considerably over the past few months, and I have hopes of reducing my most pressing debts. I have not applied to the Amer. Scand. Found., but on the other hand, neither have I given up the thought of going to America, because I cannot go on living on scholarships, that is, on charity alone, and a tenured position is still not in view, and it is very unsatisfactory to earn just 100 crowns a month by wasting my time at the Geodetic Institute on work which any undergraduate could accomplish to perfection – such as writing out address labels for some 100 envelopes for postal delivery of seismic publications! I am throwing all my energy into differential equation with the Rockefeller in mind.*

Rasch never travelled to the United States as a mathematician, and as a matter of fact, he never made a living as a mathematician. In spite of this, he remained proud of his mathematical background. As late as 1979 he said in the Andrich-Interview:

> *That is one thing I want you to make quite clear. That is, that my old teacher, Lehn, was quite right when he declared that the son of Mr. Rasch was a born mathematician. I am a born mathematician. Not one of the best in the world, by any means, but the interest in mathematics and the need to undertake research in mathematics has followed me from very early days until my untimely death may come. So this is one point I want to stress: that, although, I have been most known as a statistician, my original training and my original gift is in mathematics.*

5 Turning to statistics

According to Rasch's statements in the Andrich-Interview, it was quite accidental that he ever came to be a statistician. The story goes as follows:[16] After Rasch had defended his doctoral thesis, an old friendship from his time at the Regensen, was renewed. Rasch met M. Fog and G.K. Stürup, who were working on a project on the physiological processes in a rabbit's brain (IMSOR, 1971, p. 33). Fog and Stürup discussed their data with Rasch. In the Andrich-Interview, Rasch said:

[16] Section 3 is based on the Andrich-Interview.

> *This is what started me in a different direction* [...] *and they showed me a picture of it. They said, "don't you think that like all other curves in physiology, this is a hyperbola?" Well I looked at it and said that hyperbola have got to have two asymptotes while here I could discover no more than one* [...] *but it could be an exponential. They thought that very interesting and suggested I might like to go over their figures and see whether I was right or not. Well I had no experience in dealing with that sort of thing. But they encouraged it as friends, and so I did it for friendships sake* [...] *That was my first experimental paper.*

A paper was published as a result of the cooperation between Fog, Stürup and Rasch, namely *Über die Resorption der Zerebrosoinalflüssigkeit*, published by *Skandinavisches Archiv für Physiologie* in 1934 (Fog, Rasch, Stürup, 1934).

Seeing Fog's and Stürup's fertile cooperation with Rasch, some 5-6 people from within the medical sciences became interested in learning basic mathematics. They asked Rasch to be their tutor. Apparently, Rasch's tutorials were successful, because afterwards, Rasch was asked to give yet another course, this time in statistics. Word got around to Louis Sigurd Fridericia, professor at the Hygienic Institute.[17] (Kjærulff og Tuxen, 1990, p. 419) In the Andrich-Interview Rasch said:

> *Then Fridericia sent for me and said to me "Well Dr. Rasch, it is a very interesting thing I heard about you. I have been told that you are giving a course in statistics." "Oh no" I said, "Don't take any notice of that." "Well" he said, "it does seem that you have given such a course." I had to admit it, of course, but I wasn't very proud of it."Well couldn't you repeat that", he said, "I would be happy to be your first student."* [...] *The main point was that Fridericia was so interested in my being interested in statistics – which I wasn't at all. Then he called me to the Hygienic Institute and asked me, "We have some data here, collected this and that way, and we don't really know what to do about these figures."* [...]
>
> *Then I entered into an agreement with Hygienic Institute as a consultant. That was my decisive step. There I could get my fingers really dirty.*

Rasch was attached to the Hygienic Institute from 1934 to 1948. In 1935 he began to work as a statistical consultant at State Serum Institute as well (Årbog, 1941-

[17] In Danish: Hygiejnisk Institut.

42. p. 39). This came about when Rasch met an old friend from his time at the Regensen, namely Claus Larsen. At this time Larsen worked at the State Serum Institute. Rasch said:

> *Then Claus and I began to talk and we had known each other fairly well at Regensen. Well, then he discovered that I was beginning to take an interest in applying mathematics to a study of figures from medicine and so on. He said, "Rasch, I think we really need you in my Department. I'll go and talk with the head of the institute about you". He did so* [...] *Then I became attached to the State Serum Institute as well as the Hygienic Institute. Then I began to have a living.*

The director of the State Serum Institute in 1935, Thorvald Madsen is very prominent in the history of the State Serum Institute, and during his time as director, the Institute achieved international renown (Lademann 11. p. 286–287). Madsen had a high opinion of Rasch and never quite stopped consulting him for statistical advice (Int. Bentzon. 27.08.1998).

6 Rasch's training in statistics

In spite of the fact that Rasch did an increasing amount work as a statistical consultant, he did not yet have any formal training in statistics.[18] All this changed when Nørlund and Madsen, who happened to be members of the same society, namely, in Danish, *Videnskabernes Selskab*, met and got talking about Rasch. Rasch said in the Andrich-Interview:

> *They met each other at an evening meeting and then they got to talk about this Dr. Rasch. They agreed that it was very good that Dr. Rasch would do a job there [at the State Serum Institute] but that he needed a proper education in the latest developments in statistics. One of them, I don't know which one, knew about R.A.Fisher. Then these two applied to the Rockefeller Foundation for a year's study for me with R.A. Fisher in London.*

Actually, Rasch was awarded two scholarships. In the spring of 1935, shortly before Rasch left for England, he was granted a Carlsberg scholarship to study three months in Oslo with Ragnar Frisch. In 1969 Frisch was awarded the first Nobel Price in economics for his contribution to the development of econometrics, but already in

[18] Section 3 is based on the Andrich-Interview.

1935 he was a well-known economist (see Lindholt og Sandberg (1965)) and Lund (1997)). However, what caught Rasch's interest was Frisch's Confluence analysis. In the Andrich-Interview Rasch said:

> *It is a method in principle related to factor analysis* [...] *I went to Oslo for a couple of months and learned the trade of it and it has fascinated me ever since. I have alternated between that and linear factor analysis as long as I thought in those terms. Now I am afraid I never use these two more or less related methods because I don't think there is anything in social sciences that is linear.*

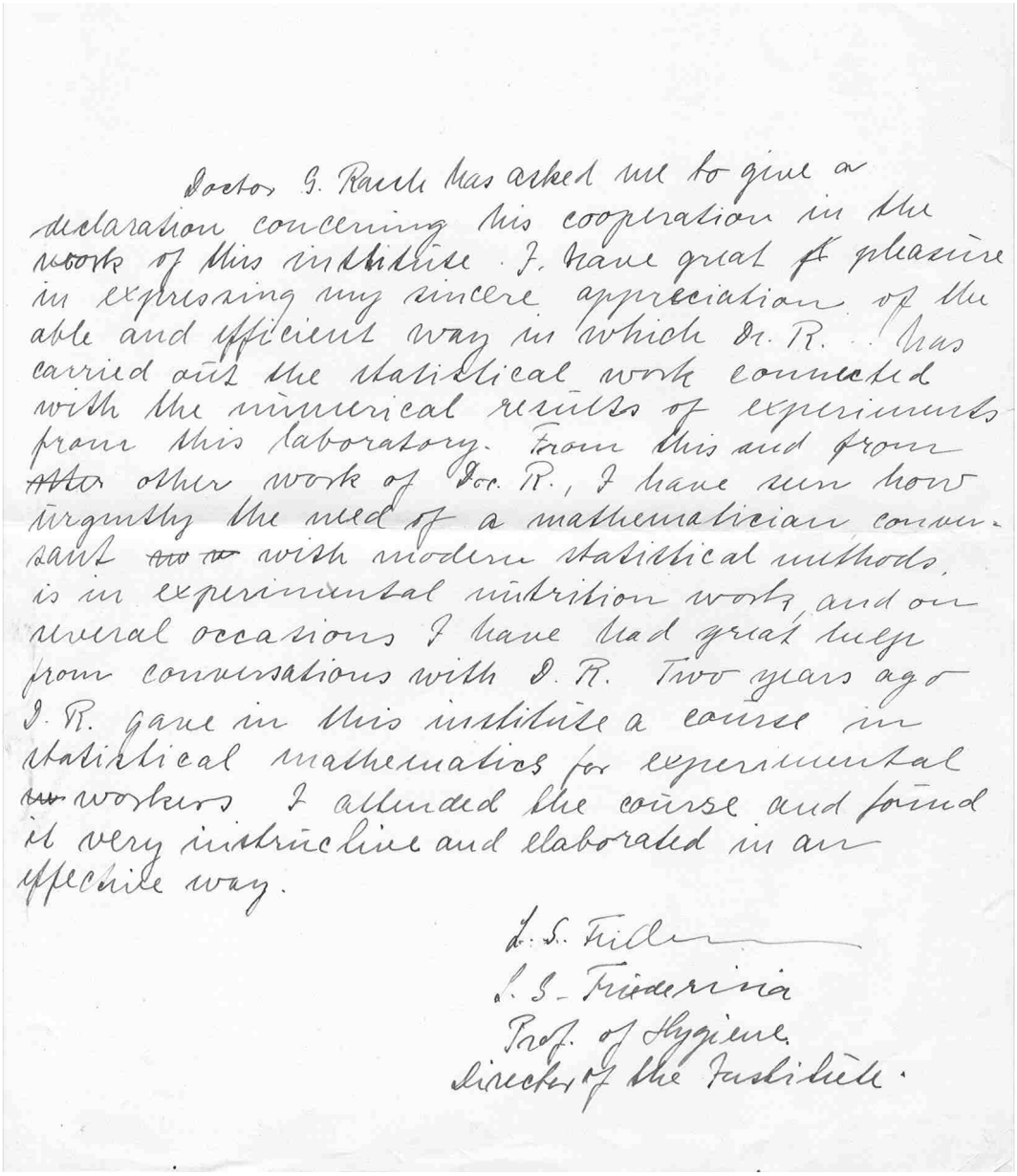
In the academic year 1935 to 1936 Rasch studied in London (Årbog, 1941-42, p. 58). During the first 8 or 9 months he studied with R.A. Fisher at the Galton Laboratory. At this time Fisher had already contributed to almost every area within the theory and foundation of statistics (Box, 1983). All his life, Rasch was very enthusiastic about Fisher, or to be more precise, about Fisher's point of view on statistics. Andersen writes in his obituary of Rasch (Andersen, 1980, p. 537–538):[19]

> *But his stay in London also moulded his own basic attitude towards statistics* [...] *and there is little doubt that his perspective on statistical theories was primarily influenced by Fisher. Well into the 1960s I heard him use quotes and examples from Fisher in connection with discussions about basic issues of statistical analysis.*

Rasch himself expressed it this way in the Andrich-Interview:

> *I learnt a lot there. Of course, I went through his statistical methods.* [...] *I got hold of his paper from 1922 where he developed the theory about maximum likelihood. I was interested in that matter* [...] *The method of least squares is not, in Fisher's interpretation, just a minimization of the sum of squares. It is the maximization of the probability of the observations, choosing such values as estimates of the parameters that will maximize the probability of the set of observations you have your disposal. There is a very essential difference from just minimizing a sum of squares. That philosophy went further when he got to the concept of sufficiency. That I think is really the high mark of what he ever did. That was the realization of the concept of sufficiency. Many may consider it just a mathematical trick, but I think it's much more than that.*

[19] The text in the quote is translated from Danish, see e.g. Appendix B for the original text



Doctor G. Rasch has asked me to give a declaration concerning his cooperation in the works of this institute. I have great pleasure in expressing my sincere appreciation of the able and efficient way in which Dr. R. has carried out the statistical work connected with the numerical results of experiments from this laboratory. From this and from other work of Doc. R., I have seen how urgently the need of a mathematician conversant with modern statistical methods, is in experimental nutrition work, and on several occasions I have had great help from conversations with D. R. Two years ago D. R. gave in this institute a course in statistical mathematics for experimental workers. I attended the course and found it very instructive and elaborated in an effective way.

L. S. Fridericia
L. S. Fridericia
Prof. of Hygiene
Director of the Institute.

FIGURE 5. A reference for Rasch written by Louis Sigurd Fridericia. A Danish version of the reference is dated the 2nd of July 1935. There exist other recommendations (in Danish) of Rasch written by T.H.Madsen and Harald Westergaard. These are from the same period in time, and they recommend Rasch as an able candidate to study mathematical statistics. It is thus likely that the above recommendation is connected to Rasch's application for a Rockefeller scholarship.

> *Any how, I feel that this is the most important thing I got from Fisher. I played quite a lot around with the concept of sufficiency.*

There is no doubt that Rasch admired Fisher very much, but taking into consideration that Rasch's work on the Rasch models and the concept of sufficiency are strongly connected, it is important to document that Rasch did not exaggerate his early interest in sufficiency. In the Andrich-Interview Rasch said:

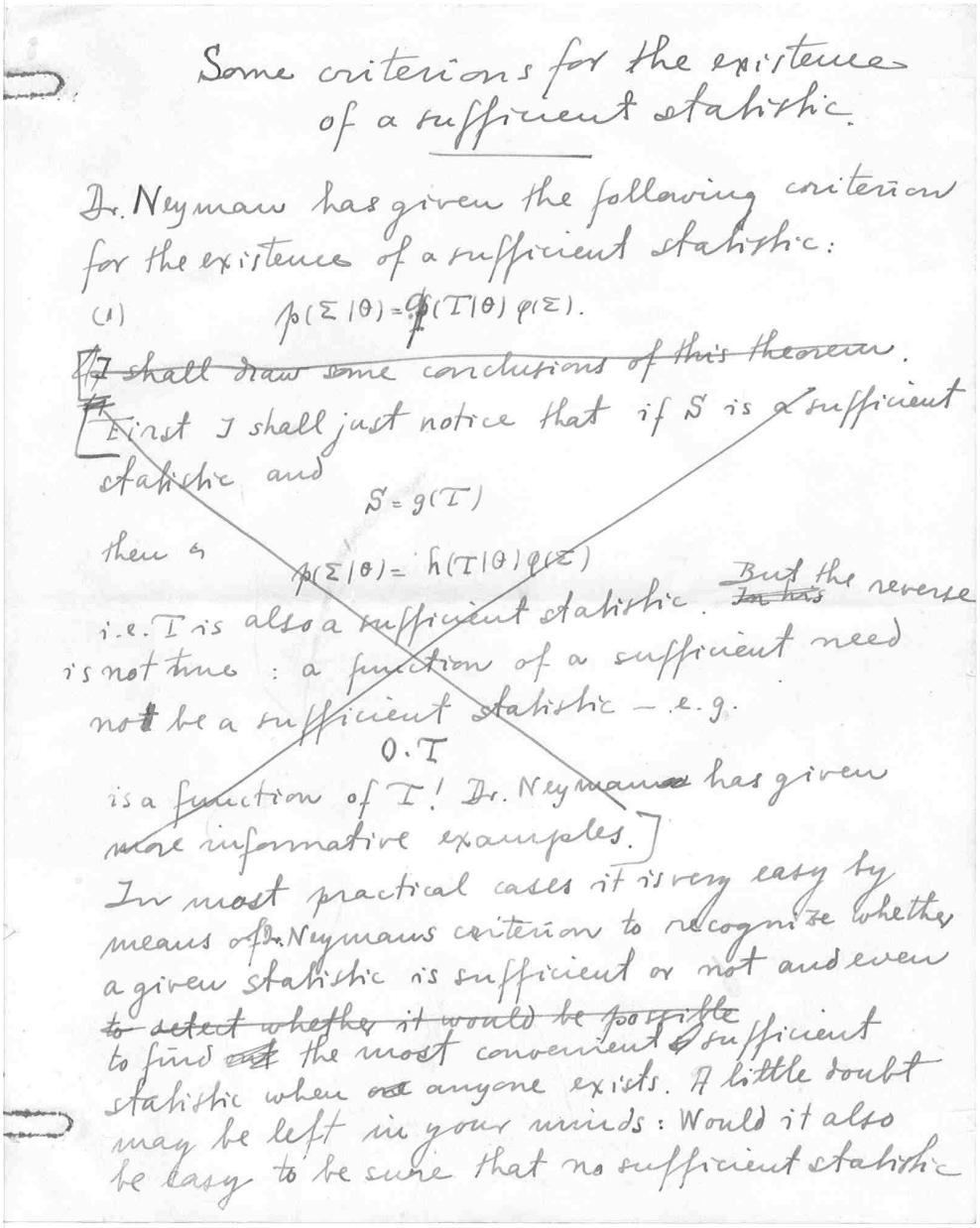
> *As a mathematician I was interested in the conditions that should be fulfilled by the probability distribution in order to allow for what Fisher called sufficient estimation. By starting from the very definition I found out the necessary and sufficient condition for that. Then I showed it to Fisher. He said "Well, Rasch that's very interesting. By the way, have you looked up the paper I wrote last year called Two New Properties of Mathematical Likelihood." I looked that up, and hidden in the derivation of the characteristic function of the probability distribution for the sufficient statistic when it exists, I found the conditions in a passage which was easily overlooked. He mentioned it just in passing.*

Actually, this paper on sufficiency that Rasch mentions he showed Fisher was found in Rasch's library. A scanning of the first page of this paper has been included here (c.f. Figure 6.). This quite substantial paper clearly demonstrates that Rasch was already interested in sufficiency while in London.

When Rasch was in London, he also met E.S. Pearson and Jerzy Neyman, who were heads of the statistical department at the Galton Laboratory. Pearson and Neyman had also contributed to the theory of mathematical statistics, but fundamentally, Fisher's perspective on theoretical statistics was very different from that of Neyman and Pearson. Rasch said in the Andrich-Interview:

> *I followed the course of lectures given by R.A. Fisher. But in the same building, Egon Pearson and Jerzy Neyman were also located. I also followed their presentations of the same theories I heard about from Fisher, but in a quite different version with their own fancy ideas. When I left London University I was familiar with both traditions of statistical methods.*

During the final 3 or 4 months in London, Rasch studied relative growth with Julian S. Huxley (Aarbog, 1941–42, p. 59). Huxley was known as a writer and a biologist, but not as a statistician (Lund, 1997). But, Rasch was analysing the growth



Some criterions for the existence of a sufficient statistic.

Dr. Neyman has given the following criterion for the existence of a sufficient statistic:

(1) $p(\Sigma|\theta) = f(T|\theta)\,g(\Sigma)$.

~~I shall draw some conclusions of this theorem.~~

First I shall just notice that if S is a sufficient statistic and

$S = g(T)$

then is

$p(\Sigma|\theta) = h(T|\theta)\,g(\Sigma)$

i.e. T is also a sufficient statistic. But the reverse is not true: a function of a sufficient need not be a sufficient statistic – e.g.

0.T

is a function of T! Dr. Neyman has given more informative examples.]

In most practical cases it is very easy by means of Dr. Neymans criterion to recognize whether a given statistic is sufficient or not and even to find the most convenient sufficient statistic when one exists. A little doubt may be left in your minds: Would it also be easy to be sure that no sufficient statistic

FIGURE 6. The first page of a paper on sufficiency dating back to 1936, when Rasch was in London.

of children at the Hygienic Institute, and Huxley had written a book on growth of animals, so Rasch was interested in learning about Huxley's theories.[20]

When Rasch left for London in the fall of 1935, Nille and their two daughters, Helga and Lotte, stayed on in Denmark. Rasch kept a lot of the letters sent to him by Nille, and from these letters it is obvious that, while Rasch was in London, Nille managed the household affairs, and that Rasch transferred funds to the family. The Rockefeller scholarship probably considered that Rasch had a family to support. The letters also tell us that Nille missed her husband, and when it was decided that Rasch would come home for Christmas, the entire family was thrilled. In a letter to Rasch, Nille wrote:[21]

> *Hurrah! You are coming home for Christmas. We are all rejoicing. Helga is happy and Lotte tells anyone who cares to listen that "Daddy will be back for Christmas!"*

After Christmas Rasch returned to London, but it was agreed that Nille and the children should join him later on. In about February Nille began the preparations for her and the children to leave the apartment in Engelsborgvej in Lyngby and travel to meet Rasch in England. Nille wrote in a letter to Rasch on the 8th of March 1936:[22]

> *My dear beloved!*
>
> *Thank you for yesterday's letter and congratulations on your apartment. Could it be that at long last we will manage to come over? We are eagerly anticipating coming and I am packing at full speed. There is a lot to do when you are leaving your home for such a long period and in such a state that others are able to use it. I haven't let our apartment yet, but I have emptied cupboards and drawers, and I have prepared lists and hope that someone will show up. I am so happy that you have rented an apartment that is not too expensive, and that there is room for us as well.*

Nille and the two girls stayed with Rasch for the rest of Rasch's stay in London.

[20] Actually, Rasch's encounter with Huxley was important. Some four years later Rasch developed a growth model to analyse the growth of animals. Later, he also used this model to analyse growth in economics. Rasch published very little on the subject (see Chapter 2 of this PhD thesis for details).

[21] The letter, which is not dated, is translated from Danish, see e.g. Appendix B for the Danish text.

[22] The text is translated from Danish, see e.g. Appendix B for the Danish text.

7 Statistical consultant

When Rasch returned to Denmark in 1936, he mainly worked as a statistical consultant.[23] In the beginning he mostly consulted at the Hygienic Institute, where he was attached as a consultant from 1934 to 1948 (Wright, 1980. p. xi) but he also did an increasing amount of consulting at the State Serum Institute, to which he was attached from 1935 to some time in the 1970s. And in the period from 1940 to 1956 he was head of the Department of Bio-Statistics.

In 1952 Rasch began working as a statistical consultant to the newly established Military Psychology Group, and he also worked as a consultant to the Danish Institute of Educational Research, established in 1955. Rasch was given a great deal of freedom to do research at these two places, which was probably why he concentrated his consulting within the field of educational statistics. This proved to have great influence on the development of the Rasch models (see Andersen and Olsen (2001) and Wright (1980) for details). Since the actual development of the Rasch models is outside the scope of this paper, no further comment will be made here on this issue. A side effect of Rasch's focus on educational statistics was, however, that he did less consulting within the medical sciences after 1955.

In addition to the consulting work already mentioned, Rasch also consulted on a more 'private' level, that is, he was hired by scientists who were granted money for statistical advice. It seems that most of the time Rasch had plenty to do (Int. Arne Jensen, 08.02.2001), and that numerous scientists consulted him. Especially those who wanted to write a doctoral thesis (c.f. the quote of Meyer on Page 36).

It is not possible to give a complete account of Rasch's work as a statistical consultant at the Hygienic Institute, or at the State Serum Institute. As is evident from a list of his publications (see e.g. Rasch (1980)), Rasch was involved in many empirical projects involving a great number of people. Michael Weis Bentzon, who started working as Rasch's assistant at the State Serum Institute in 1947, and had a working relationship with Rasch at the State Serum Institute until the 1960s, describes the normal consulting procedure during his time working with Rasch:

When a client at the State Serum Institute was in need of statistical advice, a meeting with Rasch was arranged. Often, the client was a medical practitioner, planning to write a doctoral thesis. At the first the hypotheses the client wanted to

[23] Section 3 is based on the Andrich-Interview, Andersen (1980), DBL (p. 643).

examine were usually discussed, and, if the data had not yet been collected, Rasch would give guidelines as to how the experiment should be carried out. In fact, most people at the State Serum Institute followed Rasch's suggestion to consult him before the data were collected; this way a lot of unnecessary problems with regards to the design of the experiments were prevented. Bentzon says (Int. Bentzon, 27.08.1998):

> *Sometimes, the data had been collected. And then we had to tell people that it might be a good idea to talk to us beforehand, because often, we had to say that the design of the experiment imposed narrow bounds as to what information we could gain from it.*
> *But, it was clearly Rasch's idea that we should go over the problems beforehand. And, when we got the material he had this slogan, "One must draw before one can reckon."*[24]

When Rasch and Bentzon got the data, the statistical analysis could begin. True to Rasch's slogan, "One must draw before one can reckon", numerous graphs and plots were made as a starting point. When these graphs and plots eventually revealed the structure of the data, calculations were made. Bentzon says that it was not uncommon that the client had to wait a long time before he got the results of the analysis. There were several reasons for this; first of all, it must be remembered that at the time, that is before 1950, little mechanical help was available to aid the calculations, not to mention that every graph had to be hand drawn. Second, not as many statistical models nor as many statistical methods were available as there are today. And finally, Rasch had a habit of wanting to develop his own statistical methods rather than consult the literature for a solution. According to Bentzon (Int. Bentzon, 27.08.1998):[25]

> *Rasch always wanted to solve the problems on his own. He did not care much for The Literature* [...] *But Rasch always started by taking up the problems and solving them right from the beginning. I believe that I learned quite lot from his attitude but it did seem to take a frightfully long time.*

Actually, there is general agreement among people working with Rasch (Int. Allerup, 26.02.1998), (Int. Bentzon. 27.08.1998), (Int. Arne Jensen, 08.02.2001) that he seldom read scientific papers. But, as Bentzon points out, Rasch developed his own tools to analyse whatever data set he was confronted with. This trait of scientific in-

[24] In Danish: Man må tegne før man kan regne. The old saying "one must crawl before one can walk" but shifted to draw and reckon. In Danish it rhymes.

[25] The interview was in Danish, see e.g. Appendix B for the Danish text

dependence seems to have been of importance in his later development of the Rasch models.

With respect to consulting Rasch, it did cause some problems that the analyses were often delayed. Bentzon says (Int. Bentzon, 27.08.1998):[26]

> *He had many good friends at the State Serum Institute; both on the governing body, and in the various sections. There were quite a few who valued him highly, but there were also quite a few who certainly did not like that everything took such a long time.*

It was often the case when Rasch had given statistical advice that the client published a thesis or a paper afterwards, and often Rasch's work was published as a (separate) part of the client's publication. A list of Rasch's papers (see Appendix B) includes some 50 experimental papers. Most of them deal with data of a medical or biological nature. These papers are results of Rasch's work as a statistical consultant.

At the State Serum Institute, Rasch arranged seminars. The seminars were organized such that the participants would take turns to give lectures. Mostly, the subject of the seminars was the application of statistics to specific biological problems. The participants came from many different departments at the State Serum Institute and from outside the Institute. The seminars were beneficial for all (Int. Bentzon, 27.08.1998).

Bentzon was 'the last' of Rasch's assistants at the State Serum Institute and when Rasch began to concentrate his consulting within the field of educational statistics, Bentzon took over Rasch's position. The first assistant Rasch employed was Anders Hald. This was in 1939. Other assistants followed, amongst them was Arne Jensen (Int. Arne Jensen, 08.02.2001). Both Hald and Arne Jensen later became professors of statistics. Bentzon says about being an assistant for Rasch (Int. Bentzon, 27.08.1998):[27]

> *That which I appreciated the most was his personal instructions and our discussions. We had many discussions about theory and we shared the enjoyment of solving the problems* [...]
> *I might disagree with Rasch, and then he could be sharp, but there was*

[26]The interview was in Danish, see e.g. Appendix B for the Danish text
[27]The interview was in Danish, see e.g. Appendix B for the Danish text.

> *never any contempt involved from his side. I knew Rasch in private life, and saw his effort on this account. And, actually, this meant a great deal in our relationship. The private life is important.*

Even though Rasch had turned towards statistics, he still maintained a link to mathematics, and he still lectured on mathematics at the University of Copenhagen as Nørlund's assistant in the late 1930s. Henrik Meyer, who started his study as a mathematician in 1929, wrote about Rasch in the late 1930s (Meyer, 2000, p. 13):[28]

> *Nørlund had a research assistant, Dr. Phil G. Rasch, who had written a thesis about the use of matrix algebra on differential and difference equations. He had announced tutorials in some function theoretical subject or other, I have forgotten which, and I wanted to attend to find out whether this was something for me. Therefore, I turned up the first time out of curiosity and then it turned out that there were no other students. Rasch, therefore, suggested, very sensibly, that it was not fair to give lectures. But that we could start to investigate a few problems which he was trying to solve himself. That went on for three semesters* [...]
>
> *As a mathematician Rasch was very resourceful but also somewhat messy, and very often miscalculations appeared in what we had worked out. He lived in Holte [North of Copenhagen] and therefore went to town by train every day. If he was late he did not apologize, but laughed hugely and boisterously, saying: "Isn't it just a glorious illustration of quantum theory, it is only possible to arrive late by integers of 20". Rasch had plunged into statistics and he was particularly interested in applying this discipline to the medical sciences. He was working at The State Serum Institute, and he was a terror to medical doctors when he appeared at their vivas and criticized their use of statistics. The clever guys used him instead as a consultant. Then all eventualities were covered.*

8 Private life

Outside Rasch's life as a statistician his private life revolved around his wife, Nille, his children and their villa at Skovmindevej in Holte. The villa was quite big, with three front rooms on the ground floor, a full basement, and a full first floor.

[28] The quote is a translation from a Danish text, see e.g Appendix B for the Danish version.

Interviews suggest that Rasch was a good father and that his married life was harmonious. Bentzon says (Int. Bentzon, 27.08.1998):[29]

> *He was married, and had two daughters. I believe he was an excellent father, and I also believe they had a very harmonious relationship.*

On the other hand, according to Børge Prien, who later became Rasch's son in law, Rasch was often preoccupied with scientific problems, and hence detached from the more practical aspects of family life (Int. Prien, 08.02.1999). Instead, it was Rasch's wife, Nille, who took care of these things, especially when it came to giving parties. Arne Jensen, Rasch's assistant at the State Serum Institute in the mid 1940s, professor of statistics at the Polytechnical College from 1963, and a good friend of Rasch, says (Int. Arne Jensen, 08.02.2001):[30]

> *His wife Nille was very vivacious and she was extremely supportive. They entertained quite a lot and she was extremely good at it. She was very pleasant.*

Prien describes Nille as Rasch's Muse: little, chic and musical – and very popular among friends of the Rasch-family. She was a housewife, and she took care of the children (Int. Prien, 08.02.1999). Rasch's younger sister, Agnete Toft describes Nille as a real lady. She liked fine dresses and to have them sown by a dressmaker. She had poise, and Rasch was very proud of her (Int. Toft, 10.10.2001).

When Rasch and Nille gave a party or a banquet, Nille supervised the arrangements. She was quite good at it, and, according to Toft, Nille loved to entertain. She and Rasch had birthday's close together, and they often gave a party on this occasion. However, all sorts of parties were given, varying from very large ones, to small ones. The parties were given at their home. Sometimes Rasch and Nille would play the piano well on into the small hours (Int. Toft, 10.10.2001). These parties for Rasch's friends, colleagues and clients were important for Rasch's work as a statistical consultant (Int. Prien, 08.02.1999).

Toft also tells that Rasch and Nille had a very wide circle of friends, varying from doctors to tram conductors. They had a special gift for making their guests feel comfortable, and everybody would feel at ease. The two of them created a very beguiling home (Int. Toft, 10.10.2001).

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[29]The interview was in Danish, see e.g. Appendix B for the Danish text.

[30]The interview was in Danish, see e.g. Appendix B for the Danish text.

All of this sounds very idyllic, and presumably it was. But of course, certain aspects of Rasch's private life were less idyllic. Interviews have left the impression that two of these aspects were integral parts of Rasch's character. They were borth related to the fact that both Rasch and Nille loved sumptuous living (Andrich, 2001).

First, sumptuous living is quite expensive; a big villa, big parties, having dresses made etc.However, it seems that Rasch did not care much about the expenses. Rather than adjusting his expenses to his income, he adjusted his income to his expenses (E-mail, Ellen Andersen, 29.08.2002). However, when it came to earning the money, Rasch was often a little behind schedule (Int. Allerup, 26.02.1998), (Int. Arne Jensen, 08.02.2001), (Int. Toft, 10.10.2001).

Second, Rasch's sumptuous living involved drinking wine, sherry etc, and Rasch would get drunk. As Bentzon puts it, it was nice to see Rasch in private, but it was not so nice to see Rasch drunk. Generally speaking, Bentzon approved of Rasch, and he therefore passed this over. Bentzon emphasizes that Rasch's tendency of getting drunk never influenced his work at the State Serum Institute; it was something that only happened at parties (Int. Bentzon, 27.08.1998).

As to Rasch's every day life, the family usually followed a particular pattern; the family would dine together, then Rasch would go to his study to work. Later, Rasch would come down to play cards with Nille, while they had a glass of sherry or something like that. Afterwards, Rasch would go back to his study to work far into the night (Int. Allerup, 26.02.1998). Arne Jensen says (Int. Arne Jensen, 08.02.2001):[31]

> *Rasch was a very lively person. He only slept for five hours and he usually had several projects going simultaneously but even then, if someone asked for advice, he would offer it, age notwithstanding.*

Arne Jensen once asked Rasch how come he was so generous about spending his time helping people. Rasch had laughed and said that he had often asked himself that very question, and that he had reached the conclusion that if he helped this young man at this very moment, then, what he would otherwise have been doing, would be pushed a little bit out in the future, and consequently the next point on the agenda would also be pushed a little bit out in the future – and in this way, every job would be pushed a little, until the time of his own death would be surpassed, and then it would not matter (Int. Arne Jensen, 08.02.2001).

[31]The interview was in Danish, see e.g. Appendix B for the Danish text.

9 Rasch; an engaged scientist with stature

In an interview it was emphasized that Rasch had stature both as a scientist and as a human being (Int. Mossin, 24.09.2002). I have also been told that Rasch gave the impression of being a *real* scientist: engaged in his work and scientifically interested in various topics (Int. Groth, 19.04.2002), (Int. Toft, 10.10.2001). This Section will elaborate on these traits of Rasch's.

As to the statement that Rasch had stature as a scientist, it is hardly necessary to elaborate on this, considering the fact that he became famous for the development of the Rasch models.

As to the statement that Rasch had stature as a human being, this might need some more explanation, since this trait did not prevent Rasch from ever raising his voice against other people. On the contrary, he was said to be very rude occasionally and to have an inflexible nature (Int. Karpatschof, 23.04.2002). But, nonetheless, his stature as a human being revealed itself in the fact that he was very generous in helping other people; both when it came to people who Rasch had little previous knowledge of (c.f. Arne Jensen's statements on Page 38), and when it came to his employees and students (Int. Mossin, 24.09.2002). Interviews carried out by the present author have revealed cases, where Rasch acted on behalf of his employees and students, from time to time drawing on his wide-ranging connections. It seems that Rasch genuinely cared that these young scientist got paid and furthermore, he cared that they got a scientific push in the right direction (Int. Allerup, 26.02.1998), (Int. Arne Jensen, 08.02.2001), (Int. Mossin, 24.09.2002).

With regards to the statement that Rasch was a *genuine* scientist, Sections 3 through 6 substantiate this statement about Rasch. In particular Rasch's letters to his friend and fellow scientist, Hille, in Section 4 leave no doubt about his engagement with science. Interviews suggest that this characteristic did not fade away as the years went by (Int. Andrich, 06.02.2002), (Int. Groth, 19.04.2002). Furthermore, Rasch's scientific curiosity was not confined within his own lines of science. For a period of some 10 years he was involved in the Nordic summer schools,[32] which dealt with interdisciplinary research. Through these summer schools Rasch involved himself in various topics, and although there are clear indications that he was not

[32] In Danish: Nordisk sommer Universitet

fond of writing, his interest in the Nordic summer school resulted in at least four papers, namely Rasch (1952), Rasch (1955), Rasch (1959a) and Rasch (1959b). In the Andrich-Interview Rasch emphasized that a fruitful scientific environment was created, were scientists from various branches discussed sciences. The scientists' families were also present at the summer schools, which also had a social side, and in some sense, Rasch's wife, Nille, being very good at arranging parties, was as involved as Rasch in these activities.

It seems that Rasch generally enjoyed discussing science with other scientists. A correspondence between Rasch and the statistician, Joe Berkson substantiates this. A quote from the correspondence is presented here, as Rasch's language portrays his personality:

> *Dr. J. Berkson*
> *Division of Biometry and Medical Statistics,*
> *Mayo Clinic, Rochester, Minnesota,*
> *U.S.A.*
>
> *Dear Joe*
> *Why the hell couldn't you like a decent fellow come along to Rome or anyhow to Florence in time to catch me for a heart to heart talk about your Bellagio paper. Now you have forced me to write about it, and I who loathe writing! Talking over a beer - or over a dozen - is much more comfortable. And on the top of it I havn't even seen a glimpse of your paper and your abstract together with the sheet of formulas, although very helpful, are of course inadequate when it comes to details.*
> *Therefore, I have to ask for a large margin for misinterpretations, mistakes, etc. And furthermore, as writing requires a formulation in a very finite number of words I may happen to express myself very strongly and in a most confusing way. But just shoot back. This business is too important not to be severely discussed.*
> *After this lengthy introduction I shall try to get down to the brass tacks, dug up in a conversation with Rao and Hald, and polished in a later discussion with Bentzon* [...]

Rasch continues the letter with some 8 pages of formulas and verbal arguments. The subject of the correspondence was whether the estimator Berkson suggested in his

Bellagio paper was a sufficient statistic or not. An other four letters on the subject exist, including one that Rasch sent to C.R. Rao on the matter. It is tempting to go into the theoretical details of this correspondence, but this is outside the scope of the present paper. The important thing is that the letters portray Rasch as an engaged scientist, who approached other scientists on scientific matters. Second, it may be noted that the letter leaves the impression the Rasch saw scientific discussions as something enjoyable and that having a good time and discussing sciences were activities that went hand in hand. Finally, the correspondence further documents that Rasch was quite interested in the concept of sufficiency prior to publishing the Rasch models (c.f. Page 30 of this thesis.).

10 Communicator of Fisher's ideas

Rasch is internationally recognized for his development of the Rasch models. But even if he had not published this class of models, he would still figure in Danish books on the history of the University of Copenhagen, namely as a statistician who, at some level, influenced the progress of statistics in Denmark. This section describes the bearing Rasch had on the progress of statistics in Denmark.[33] To do so, we shall return to the time when Rasch returned from his sabbatical year spent by Fisher.

When Rasch returned from England, he was one of the people in Denmark who knew most about Fisher's new theories on mathematical statistics. At that time, statistics was a relatively new line of sciences, and it had not yet achieved a high level of technical sophistication. In 1936 there was only '1 1/3' scientists employed at the University of Copenhagen to do research in statistics, namely Hans C. Nybølle, a professor of statistics at the Faculty of Social Science, and Johann Frederik Steffensen, reader in insurance science at the faculty of Mathematics and Natural Science. Because Steffesen's time was equally divided between actuarial mathematics, numeric analysis and mathematical statistics, he only counted as 1/3 of a person with regards to doing research in statistics (Hald, 1994, p. 22). Nørlund also gave some courses in statistics because of his interest in geodesy. Anders Hald, who in 1937 was an undergraduate student, and who in 1948 became professor of statistics

[33]The details in Section 10 concerning Rasch's work as a consultant tutorials at the University of Copenhagen are based on Aarbog (1939-1940), Aarbog (1941-1942), Andersen (1980), Andersen (1986), Betænkning (1950), Gottschau (1983), Hald (1983), Hald (1994), Wright (1980), The Andrich-Interview, Årbog (1944-1945) and Årbog (1958-63).

at the University of Copenhagen, writes about the mathematical level of statistics back then (Hald, 1994, p. 23):

> *I took my first course in statistics in 1935 by Nybølle, my second in 1936 by Steffensen, and my third in the method of least squares for geodesists by N. E. Nørlund in 1936.*
>
> *The mathematics was dull, estimation meant routine calculation of point estimates by the method of least squares, there was no unifying idea, there was no enthusiasm, it was at the end of the second revolution, and the revolutionary spirit had disappeared.*
>
> *On this background you may well ask what induced me to continue studying statistics. The answer is simple: Fisher has the full responsibility. His theory came as a revelation to me, transmitted through Steffensen and Rasch.*

The first time Rasch gave lectures in statistics at the University of Copenhagen was in the spring of 1937. At this time he was still Nørlund's assistant and on Nørlund's request, Rasch gave a short series of lectures on Fisher's results on the linear normal model. Hald followed this series of lectures.

Hald describes the period after Rasch's return from England and up to 1950 as a time where Fisher's ideas on statistics began to spread to Denmark. Hald writes (Hald, 1983):[34]

> *The great communicator of Fisher's ideas was G. Rasch* [...] *Rasch exercised his influence not only through his lectures, but also through his various applications of statistics within biology, medicine and psychology and not least by allowing young actuary candidates work for short time periods as an assistant to him at the State Serum Institute.*

We have seen that Rasch had assistants at the State Serum Institute, and that Hald was the first one amongst them. As to Rasch's lectures at the University of Copenhagen before 1950, he began to give lectures on a regular basis from 1939 when a lectureship in biological mathematics was *established* for him. In 1945 this lectureship was expanded to include mathematical statistics, and this was actually the first time that mathematical statistics became an independent discipline at the University of Copenhagen (Aarbog, 1944-1945, p. 53). In the spring semester of 1939 Rasch

[34]The text has been translated from Danish, see e.g. Appendix B for the text in Danish.

also gave lectures in statistics to psychologists, this because of a vacancy (Aarbog, 1939-1940, p. 54), however, in 1944 with the establishment of an exam for psychology and educational studies, these tutorials continued on a more permanent basis.

There seems to be general agreement that Rasch was a prominent figure in the progress of mathematical statistics in Denmark, for example, Andersen writes in his obituary of Rasch (Andersen, 1980, p. 537−538):[35]

> *For mathematical statistics in Denmark, Rasch's return from London bringing the latest news from the place where modern statistics was born meant a breakthrough. Through his contribution and through that of professor Hald, his research assistant at the State Serum Institute from 1939 to 1942, Fisher's ideas came to mark the development in Denmark.*

However, as is evident from a list of Rasch's publications, he mainly wrote experimental papers in the years following his return from England, and however brilliant these might be; they were not textbook presentations for future statisticians. In fact, before he published the Rasch models, he had few publications that were influential from a theoretical or statistical point of view (IMSOR, 1971). Since Rasch was paid by the Danish Institute of Educational Research to write his book on the Rasch models, Rasch (1960), (see Andersen and Olsen (2001)) it is not a far fetched guess to say that Rasch was too busy earning money to get much research done on his own behalf.

Instead, it was Hald who wrote books on Fisher's theories on mathematical statistics, namely e.g Hald (1938), Hald (1948) and Hald (1952); the latter one being used as a text book at the University of Copenhagen, Faculty of Social Sciences when Hald was a professor there. Furthermore, while there seems to be general agreement that Rasch was an outstanding teacher when alone together with one or two students, there is also general agreement that Rasch's lectures to a larger audience could have been better prepared. Bentzon has said (Int. Bentzon, 27.08.1998):[36]

> *Face to face he was magnificent and as you can easily understand we had a great time together, and I am sure I learnt a lot from him* [...] *Unfortunately, I have to add that as a lecturer I do not think he was particularly gifted, because he had a tendency not to be thorough, and his lectures did not seem to be properly worked through. I do not hope I am being unfair*

[35]The text has been translated from Danish, see e.g. Appendix B for the text in Danish.
[36]The interview was in Danish, see e.g. Appendix B for the Danish text

> [...] *Nonetheless, it was easy to see when you attended his lectures that his was a new perspective on the subject.*

In conclusion, there is no doubt that Rasch was a prominent figure in developing mathematical statistics in Denmark. However, his seems mostly to have been as a mentor to the new generation and that he brought to them new ideas on statistics.

11 Summa Summarum

Georg Rasch was born in 1901 and he died at the age of 79 in 1980. As a scientist he became well known for developing the statistical models often referred to as *the Rasch models*.

In his youth Rasch studied mathematics, and throughout his adult life he defined himself as a mathematician. In Denmark, however, there was no available positions for Rasch in mathematics, and he turned to statistics. He was granted a Rockefeller scholarship to spend a sabbatical year with R.A Fisher at the Galton Institute in London.

From the early 1930's until 1960, Rasch's main source of income came from his work as a statistical consultant. In the beginning he was mostly consulted by the medical sciences, but from the mid 1950's his consulting became concentrated in the field of psychology and he did a lot of work with educational statistics.

In his empirical work Rasch tended to develop his own methods instead of searching the literature for a solution. He was an independent thinker, which most certainly was an asset for his later development of the Rasch models.

One of Rasch's accomplishments prior to the development of the Rasch models was to influence the progress of statistics in Denmark; not so through the lectures he gave at the University of Copenhagen, but more so because he influenced and inspired his young students.

Rasch seems to have been a scientist of stature; deeply engaged in science and at the same time generous in his efforts to give young scientists an opportunity.

12 Appendix A. Interviews

Appendix A contains background information about the people who so kindly told the present author about their memories of Rasch. Special emphasis is on their connection to Rasch.

Allerup, Peter. Interview 26.02.1998. Peter Allerup is a statistician by training. He became associated with Rasch in the 1960s, and he worked with him at both the Statistical Institute and the Danish Institute of Educational Research, where Rasch was affiliated as a consultant. Allerup has continued to work with the Rasch Models at the Danish University of Educational Research, where he now holds a position as professor of statistics.

Ellen Andersen E-mail 29.08.2002. Ellen Andersen is an economist by training. She was employed at the Institute of Economics in 1965, and in 1973 Ellen Andersen was appointed professor of empirical economics. Rasch was a professor at the Statistical Institute from 1962 to 1971, and the intention was that statistics was to be an auxiliary subject for economics and sociology. Hence, Ellen Andersen came to know Rasch through their work and cooperation together at the same faculty. (see. e.g. Chapter 4 of this thesis).

Andrich, David. Interview. 04.02.2002 and 06.02.2002, Perth, Western Australia. Andrich met Rasch in 1972 in Chicago. At this point in time, Andrich was a graduate student, and Rasch was newly retired. Rasch was visiting Ben Wright in Chicago, and Wright, who was the Chair of Andrich's PhD committee, persuaded Andrich that, he on his way back to Australia in 1973, he should visit Rasch in Denmark. Andrich stayed with Rasch for a week or so, and in 1974 Andrich arranged for Rasch to be a visiting professor for seven months in the Departments of Mathematics and Education at the University of Western Australia, where Andrich held a position. Andrich visited Rasch again in 1975, and in 1977 Andrich spent five months at the Danish Institute of Educational Research, where Rasch was still affiliated as a consultant. In 1979, Andrich visited Rasch with the purpose of interviewing him. The interview Andrich conducted on this occasion is the one referred to throughout this article as the Andrich-Interview. Andrich is now professor of Education at Murdoch University in Western Australia.

Bentzon, Michael Weis. Interview 27.08.1998. Bentzon got to know Rasch in the late 1940s, when Bentzon attended the course Rasch gave on statistics at the

Faculty of Mathematics and Natural Sciences. Bentzon thereafter came to work for Rasch at the State Serum Institute, where Rasch worked as a statistical consultant. Their working relationship continued throughout the period Rasch was with affiliated the State Serum Institute, that is, until the late 1960s. Bentzon and Rasch also saw each other socially.

Groth, Christian. Interview 19.04.2002. In the early to mid 1960s Groth was a student of sociology, and as such, he attended Rasch's lectures on statistics in the mid 1960s. He later became an instructor at the Statistical Institute, where Rasch was a professor of statistics. Today, Groth is an economist and he is an associate professor at the Institute of Economics.

Arne Jensen Interview 08.02.2001. Arne Jensen became associated with Rasch in the early 1940s when he began to work for Rasch at the State Serum Institute. In 1963, Arne Jensen was appointed professor of statistics at the Polytechnical College (today the Danish Technical University). Arne Jensen was a good friend of Rasch throughout Rasch's life, and they saw each other socially. Both Arne Jensen and Rasch had weekend cottages on the Danish island of Læsø.

Karpatschof, Benny. 23.04.2002. As a student of psychology, Karpatschof attended the lectures Rasch gave on statistics for psychologists (c.f. Page 10). Later, Karpatschof was employed at the Military Psychology Group and the Danish Institute of Educational Research where Rasch consulted, and it was here that Karpatschof got to know Rasch. Today Karpatschof is an associate professor at the Institute of Psychology, University of Copenhagen.

Mossin, Axel. Interview 24.09.2002. As a student of economics, Mossin attended the lectures Rasch gave as professor of statistics (c.f. Chapter 4 of this thesis). When Mossin graduated in 1965 Rasch employed him at the Statistical Institute, where Rasch held a position as a professor. Some years later, Mossin got a position at the Institute of Economics. Today, Mossin is an associate professor at the Institute of Economics.

Prien, Børge. Interview 08.02.1999. As a student of psychology, Prien attended Rasch's lectures on statistics for psychologists (c.f. Page 10). After graduating Prien did his military service by working at the Military Psychology Group. At this time Rasch was affiliated with this place as a consultant. At this point in time Rasch was developing the Rasch models, and he needed some new attainment tests to be created. Prien was the one who created these tests.

In 1956 Rasch head hunted Prien for a position at the Danish Institute of Educational Research, where Rasch was also affiliated as statistical consultant. Prien was married to Rasch's youngest daughter, Lotte, and therefore knows a lot about Rasch in private life.

Toft, Agnete. Interview 10.10.2001. Toft is Rasch's youngest sister. After the death of Rasch's mother in 1920, Rasch's father left his position as first head master in Svendborg, and some years later he married Sørine Cathrine Rasmine *nee* Sørensen, always called Kate by the Rasch family. In 1923 their daughter, Agnete, was born. As siblings, Agnete and Rasch had a close relationship, and saw each other often in adult life.

13 Appendix B. Quotes in Danish

This Appendix contains all quotes that were originally in Danish and that have been translated to English in this paper.

The original text to the quote on Page 20: *Du spørger til mine Fremtidsplaner. Ja, dem jeg havde, er røget i Lyset, saa nu har jeg strengt taget ikke nogen. Hermed forholder det sig saaledes: Da det rygtedes, at Bohr under alle Omstændigheder vilde forlade Polytkeninsk Læreanstalt, saa jeg heri mit Livs store Chance. Det maatte anses for temmelig givet, at A.F.Andersen skulde være Bohrs Efterfølger, men saa skulde han jo have en Efterfølger som Docent ved Landbrughøjskolen, og det var dette, jeg tog sigte paa, da jeg tog mig selv i Nakken og i Løbet af syv Maaneder skrev og indleverede min Disputats – forøvrigt uden at have gjort større Forarbejder. En Ide som jeg af og til havde syslet med, nemlig* [...] *– jeg lod Isotoni være Isotoni og styrtede mig paa Hovedet i de nye Undersøgelser, hvis Analogi med mine tidligere Studier over Differensligninger blev Aarsag til, at disse genoptoges og efterhaanden resulterede i Kap. III. Dog kun under ret svære Fødselsveer, for lige som jeg skulle til at renskrive det, gik det – i løbet af 8 dage – til min Forfærdelse op for mig, at jeg paa et afgørende Punkt havde tilladt mig at deformere en retliniet Integrationsvej til en nærmere angivet lukket Kurve – som imidlertid aldrig eksisterer! Dette var i begyndelsen af oktober. Mit Arbejde brød helt sammen for mig, jeg maatte pausere i ca. halvanden Maaned, men ved endnu en Kraftanstrengelse lykkedes det mig at faa Afhandlingen færdig og afleveret til Nørlund inden Juleferien. Samt under udøvelse af megen Diplomati og Overtalelseskunst at bibringe ham den Opfattelse,*

at jeg nødvendigvis maatte disputere inden Sommerferien. Det lykkedes ogsaa at faa realiseret Programmet trods mange Vanskeligheder.
Og alt det var sat i Scene af Hensyn til det skaldede Docentur – som jeg saa naturligvis ikke fik! Der var nemlig i Mellemtiden opdukket en ny Stjerne af en ifølge Bohr ganske overordentlig Lysstyrke: Dr. Børge Jessen [...] *Ved siden af denne Stjernesol blegnede jeg naturligvis ganske – og jeg har jo for Resten heller ikke Direktøren paa Polyteknisk Læreanstalt til Svigerfader.*
Imidlertid, Stillingen er ham vel undt - naar jeg nu bare vidste, hvad jeg selv skal tage mig til. (Brev fra Rasch til Hille, Rønnebæksvej 6, København, Brønshøj. d. 22-9-1930).

The original text to the quote on Page 24: *For fortsætte, som jeg har levet de sidste 3 Aar er jeg ikke meget opsat paa! Jeg har rent ud sagt 200 Kr. om Maaneden at 'leve' af (Assistent for Nørlund og et Par Timers dagligt Arbejde paa Geodætisk Institut) og har jo oven i Købet begaaet den tilsyneladende Daarskab at gifte mig. Denne Tingenes Tilstand kan selvsagt ikke vedblive at bestaa, og det er netop nu ved at brænde helt sammen for mig. Men paa den anden Side værger jeg mig ved at søge en Stilling indenfor Skolevæsenet eller lignende, da jeg frygter, det vil gaa alt for meget ud over mit videnskabelige Arbejde.* [...] *Du har naturligvis forlængst gættet, at disse Jerimiader har en ganske bestem Adresse – for ellers ville jeg virkelig ikke plage dig med dem! Ærlig talt! Er der ikke noget for en hvid Mand at gøre i USA?* [...] *Er der et eller andet sted i USA en Stilling for en haabefuld ung skandinavisk Matematiker? Og vil du i bekræftende fald søge at hjælpe mig til den? Undskyld min Paagaaenhed! Jeg indser, at mit Forlangende maaske er for stort, men det er den eneste Chance for at redde min Fremtid som Videnskabsmand, jeg for Tiden kan øjne! Jeg er dyb pessimist med hensyn til Mulighederne herhjemme.* (Brev fra Rasch til Hille, Rønnebækvej 6, København, Brønshøj d. 22.9.1930).

The original text to the quote on Page 24: *Nogen Anerkendelse – udover smukke Ord – har jeg til min store Glæde ogsaa begyndt at faa herhjemme, idet Nørlund, Niels Nielsen og Hjemslev har udvirket, at jeg har faaet tildelt et Universitetsstipendium paa 1000 kr. aarligt i 2 Aar (kan udvides til 4 Aar) samt en Understøttelse paa 1200 kr. fra Carlsbergfondet, foreløbig bevilget paa et Aar, men hvis jeg skikker mig vel, kan den fornys* [...] *Saaledes staar Sagerne da i øjeblikket: Jeg er væsentlig bedre økonomisk stillet end for et par Maaneder siden og kan gøre mig Haab om at mindske den mest trykkende Gæld; jeg har ikke søgt Amer.Scand.Found., men har heller ikke opgivet Tanken om Amerika, for dels kan jeg ikke blive ved at leve paa Stipendier d.v.s. paa Naade og Barmhjertighed, og passende Embede er stadig ikke at øjne, og*

dels er det meget lidt tilfredsstillende for mig stadig at skulle tjene 100 Kr. maanedligt ved at spilde min tid på Geodætisk Institut paa Arbejde, man end ikke behøver at være Stud.mag, i matematik for at udføre til fuldkommenhed – som f.ex. at skrive Adresser paa nogle 100 Konvolutter til udsendelse af seismiske Publikationer! Jeg arbejder derfor af saavidt mulig alle Kræfter paa Differentialligningerne med Rockefeller for Øje. (Brev fra Rasch til Hille, Sonnerupvej 30, København, Brønshøj d. 10.02.1931)

The Danish text to the quote on Page 28: *Men også for hans egen grundholdning til statistik var Londonopholdet af afgørende betydning.* [...] *Der er dog ingen tvivl om, at det primært var påvirkningen fra Fisher, der fik betydning for hans syn på statistikkens teori. Langt op i 60'erne har jeg oplevet ham citere udtalelser og eksempler fra Fisher i forbindelse med diskussion af statistiske grundlagsspørgsmål.*

The Danish text to the letter on Page 32: *Hurra! du kommer Hjem til Jul! Hvor er vi alle tre glade. Helga er lykkelig og Lotte fortæller alle og Enhver der gider høre derpaa "Far kommer Hjem til Jul!"*

The Danish text to the letter on Page 32: *Min egen Elskede!*

Tak for Brev i Gaar! og til lykke med Lejligheden. Skal det mon nu endelig blive til noget at vi kommer ? Vi glæder os umaadeligt, og jeg pakker for fuld Kraft. Der er meget at gøre, naar man skal forlade Hjemmet saalænge og forlade det saadan at andre eventuelt kan bruge det. Lejligheden er endnu ikke lejet ud, men jeg tømmer Skabe og Skuffer og tæller alt op og haaber paa at Nogen vil vise sig.
Jeg er glad for at du har faaet en Lejlighed, der ikke er saa dyr og den kan sagtens være stor nok til os.

The Danish text to the quote on Page 34: *Nogen gange så var det altså indsamlet på forhånd. og så kunne vi jo fortælle folk, at det nok var en god ide, at snakke med os først, for mange gange måtte man simpelthen sige, at den måde materialet var indsamlet på satte snævre grænser for, hvad man kunne få ud af det.*
Men det var helt klart Raschs ide, at man skulle gennemgå problemerne først. Og når man så fik materialet så havde han det der slogan, "man skulle tegne før man kunne regne"

The Danish text to the quote on Page 72: *Rasch ville altid løse problemerne selv. Han dyrkede ikke litteratur ret meget* [...] *Men Rasch startede altid med at tage problemerne og løse dem selv fra bunden. Det syntes jeg nok, jeg lærte en hel masse af, men det tog jo samtidig en frygtelig tid. Hæ , hæ.*

The Danish text to the quote on Page 35: *Han havde mange gode venner på seruminstituttet. Både blandt lederne og på de forskellige afdelinger. Der var nogen, der*

satte meget stor pris på ham, men så var der altså også nogen, der var meget kritiske over for, at alting tog så lang tid.

The Danish text to the quote on Page 35: *Det, som jeg især havde meget ud af, det var hans indsats som lære på tomandshånd. Vi havde mange diskussioner om teori. Vi havde begge det, at vi kunne more os over løsningen på problemerne. Vi havde nogle utroligt morsomme oplevelser.*[...] *Man kunne godt være uenig med Rasch, og så kunne han godt være skrap, men ikke noget med at..., altså ingen foragt var der med ham. Nu skete der jo det for mit vedkommende, at jeg omgik også med Rasch privat, og så hvordan han .. altså hans indsats i sit hjem. Og det spillede faktisk en stor rolle for mig i relation til ham. Det private er faktisk ikke helt ligegyldigt.*

The original text to the quote on Page 36: *Nørlund havde en assistent, dr.phil.G.Rasch, der havde disputeret om anvendelse af matrixregning på teorien for differential- og differensligninger. Han havde annonceret øvelser i et eller andet funktionsteoretisk emne, jeg husker ikke længere hvad, og jeg havde lyst til at se, om det var noget for mig. Derfor mødte jeg op første gang sådan af nysgerrighed, og så viste det sig, at der ikke var andre tilhørere. Rasch foreslog derfor meget fornuftigt, at det ikke var rimeligt at holde forelæsninger, men at vil i stedet kunne tage hul på at løse nogle problemer, som han havde hjemme i skuffen. Det kørte så i 3 semestre, og af dette kom så min store opgave til magisterkonferens til verden* [...] *Rasch var en iderig matematiker, men lidt af et rodehoved, og der viste sig ofte regnefejl i det, som vi havde produceret. Han boede ude i Holte og kom derfor hver dag ind med S-toget. Kom han for sent, så sagde han ikke pænt undskyld, men slog en høj latter op og udbrød: "Er det ikke en strålende illustration på kvanteteorien: Man kan kun komme et helt antal gange 20 minutter for sent."*

Rasch havde da kastet sig over statistik, og var navnlig interesseret i denne disciplins anvendelse i medicinen. Han var ansat ved Seruminstituttet, og han var lægernes skræk, når han dukkede op ved deres disputatser og kritiserede deres anvendelse af statistik. De kloge brugte ham i stedet som statistisk konsulent. Så var misteltenen taget i ed.

The Danish text to the quote on Page 37: *Hans kone, Nille, var en meget livlig sjæl og en fantastisk støtte for ham. De holdt fest for venner og bekendte og trak folk sammen. Hun var god til at holde selskaber. Hun var meget tiltalende.*

The Danish text to the quote on Page 37: *Han havde en kone og to døtre. Jeg mener han var en udmærket far. Det var på sin vis et meget harmonisk liv han havde der.*

The Danish text to the quote on Page 38: *Rasch var et meget livligt menneske.*

Han sov kun 5 timer i døgnet. Han havde som regel mange projekter igang, men alligevel, hvis der var et menneske, der spurte ham til råds, gav han hjælp, uafhængigt af alder.

The Danish text to the quote on Page 42: *Den store formidler af de Fisher'ske ideer var G. Rasch* [...] *Rasch udøvede sin indflydelse ikke alene gennem sine forelæsninger, men også gennem sine talrige praktiske arbejder indenfor biologi, medicin og psykologi og ikke mindst derved, at en række unge aktuarkandidater i kortere perioder virkede som assistent hos ham på Seruminstituttet.*

The Danish text to the quote on Page 43: *For dansk matematisk statistik betød Rasch hjemvenden fra London med sidste nyt fra det sted, hvor den moderne statistik blev skabt, et gennembrud. Både gennem hans egen indsats og gennem prof. Hald, der var hans assistent på Seruminstituttet 1939–42, kom især Fisher's tanker til at præge udviklingen i Danmark.*

The Danish text to the quote on Page 43: *På tomandshånd, der var han pragtfuld. Og som du jo kan forstå, så morede vi os samtidig, og jeg synes, jeg lærte en masse af ham* [...] *Desværre må jeg sige, at som forlæser synes jeg ikke han var god, fordi han havde det med ikke at forberede sig ordentligt. Hans forelæsninger var ikke ordentligt gennemarbejdede. Jeg håber ikke jeg gør ham uret.* [...] *Ikke desto mindre kunne man godt se, når man gik til hans forelæsninger, at det var en ny måde at se tingene på. Så på en eller anden måde inspirerede de alligevel.*

14 References

Aarbog (1924–1925). *Aarbog for Københavns Universitet, Kommunitet og Den Polytekniske Læreanstalt indeholdende Meddelelser for det akademiske Aar 1923–24.* Redigeret af H. Munch-Petersen. København.

Aarbog (1939–1940) *Aarbog for Københavns Universitet, Kommunitet og Den Polytekniske Læreanstalt indeholdende Meddelelser for det akademiske Aar 1939-1940.* Redigeret af Poul J. Jørgensen. København.

Aarbog (1941–1942). *Aarbog for Københavns Universitet, Kommunitet og Den Polytekniske Læreanstalt indeholdende Meddelelser for det akademiske Aar 1941-1942.* Redigeret af Olaf Waage og G.C.C. Ulrich. København.

Andrich, D. (2001). Understanding Resistance to the Data-Model Relationship in

Rasch's Paradigm: A Reflection for the Next Generation. Invitational presentation at a conference honouring the 100th anniversary of the birth of Georg Rasch, Copenhagen, September, 2001.

Andersen, Einar (1983). Geodæsi. In Mogens Phil m.fl. (Eds.) *Københavns Universitet 1479-1979, bind XII: Det matematisk-naturvidenskabelige Fakultet, 1. del.* G.E.C. Gads Forlag, København, 237−246.

Andersen, Erling, B. (1980). Georg Rasch. In *Universitets Årbog 1980-81,* 536−540.

Andersen, Erling, B. (1986). Georg Rasch. In Samuel Kotz and Norman L. Johnson (Eds.) *Kotz-Johnson. Encyclopedia of Statistical Sciences, Volume 7.* John Wiley and Sons, Inc. Canada. p. 626−627.

Andersen, Erling, B. (1995).What Georg Rasch would have Thought about this Book. In Gerhard H. Fischer, Ivo W. Molenaar (Eds.), *Rasch Models.* Springer-Verlag, New York, 383−390.

Andersen, Erling and Olsen, Lina Wøhlk (2001). The life of Georg Rasch as a Mathematician and as a Statistician. In Anne Boomsma, Marijtje A. J. van Duijn, Tom A.B.Snijders (Eds.).*Essays on Item Response Theory.* Lecture Notes in Statistics. Springer-Verlag New York, Inc.

Bang, Thøger (1988). Niels Erik Nørlund in Memoriam. *Acta Mathematica,* vol 161, p. 11-22.

Betænkning (1950). *Betænkning af 22 juni 1950 angående statistikkens stilling som selvstændig videnskab og som hjælpevidenskab ved de højere læreanstalter i København* (19 pages). Anders Hald gave these pages to Lina Olsen as a typewritten copy.

Bohr, Harald (1931). Niels Nielsen. 2 December 1865 – 16 September 1931. *Matematisk tidsskrift,* B, 41−45).

Box, Joan Fisher (1983). Fisher, Ronald Aylmer. In Samuel Kotz and Norman L. Johnson (Eds.). *Kotz-Johnson. Encyclopedia of Statistical Sciences,* Volume 3. John Wiley and Sons, Inc. Canada. 103−111.

DBL. *Dansk Biografisk Leksikon, Bind 1-16* Redigeret af S.C. Bech. Tredje udgave. Gyldendal, København 1979-84.

Festskrift (1930). *Festskrift udgivet af Københavns Universitet i Anledning af Universitetets Aarsfest*. November 1930. K.K.K. Lundsgaard. Gamle og nye metoder til udførelse af de almindeligste operationssnit gennem øjets væg. København MCMXXX Bianco Lunos Bogtrykkeri A/S.

Fog, M., Rasch, G. and Stürup, G. (1934). Über die Resorption der Zerebrospinalflüssigkeit. *Skandinavischen Archiv für Physiologie,* Vol 69, 127−150).

Gottschau, Adam (1983). *En analyse af et objektivitetsbegreb. Med udgangspunkt i Georg Rasch's psykometriske arbejder*, Danmarks Pædagogiske Institut. Denmark.

Hald, Anders (1939). *En matematisk Fremstilling af R. A. Fisher's Teorier*. Besvarelse af Universitets Prisopgave i Matematik for Aaret 1938. Duplikeret i 1945.

Hald, Anders (1948). Statistiske Metoder med Eksempler på Anvendelse indenfor Teknikken. Det Private Ingeniørfond. København.

Hald, Anders (1952). *Statistical Theory with Engineering Applications*. John Wiley and Sons, New York and Chapman and Hall, London.

Hald, Anders (1983). Statistikkens teori. In Mogens Phil m.fl. (Eds.) *Københavns Universitet 1479-1979, bind XII: Det matematisk-naturvidenskabelige Fakultet, 1. del.* G.E.C. Gads Forlag, København, 213-228.

Hald, Anders (1994). The Three Revolutions in Statistical Inference and Their Importance for my Life as a Statistician. In *Three Contributions to the History of Statistics.* Institute of Mathematical Statistics University of Copenhagen. Preprint.

Hille, E. and Rasch, G (1928). Über die Nullstellen der unvillständigen Gammafunktionen $P(z, \rho)$. II. *Matematische Zeitschrift,* Vol. 29, 319−334.

IMSOR (1971). *Professor, dr. phil. Georg Rasch. Oversigt over væsentligere arbejder. Udgivet i anledning af hans 70 års dag. 21. 09. 71.* Udgivet af IMSOR. (IMSOR – Institut for Matematisk Statistik og Operationsanalyse).

Kjærulff, J. and Tuxen, O. (1990). Københavns Universitet 1479-1979, bind XIV:

Universitets- og videnskabshistorisk bibliografi. Redigeret af Svend Ellehøj m.fl. G.E.C. Gads Forlag, København.

Lademann 11 (1973). *Lademanns leksikon* Redigeret af cand. mag. Torben W. Langer. Vol. 11. LARV–MALS. Lademanns Forlagsaktieselskab, København.

Lindholdt, T. and Sandberg, B. (1965). Professor dr. philos. Ragnar Frisch 70 år. *Sosialøkonomen Utgitt av Sosialøkonomisk Samfunn. 2. feb. 1965.* Specialenummer til professor Dr. philos. Ragnar Frisch i anledning hans 70-års dag 3. marts 1965.

Lund, Jørn (1997). Den store Danske Encyklopædi, Bind 7. Redigeret af Jørn Lund m.f., Danmarks Nationalleksikon Gyldendal, København.

Meyer, Henrik (2000). Nogle Erindringer fra mine læreår. *Matilde – Nyhedsbrev for Dansk Matematisk Forening. Nummer 4, Maj 2000,* København.

Nielsen, Niels and Rasch, Georg (1923). Notes Supplémentaires sur les Equations de Lagrange. *Det Kgl. Danske Videnskabernes Selskab. Matematisk-fysiske Meddelelser. V, 7,* 1923.

Nørlund, Niels Erik (1926). J.L.W.V. Jensen. *Matematisk tidsskrift*, B, 1−7. Copenhagen.

Ramskov, Kurt (1995). *Matematikeren Harald Bohr.* Institut for de eksakte videnskabers historie. Det naturvidenskabelige Fakultet. Aarhus Universitet.

Rasch, Georg (1927). Beitrag zur Theorie der unvollständigen Gammafunktionen. *Det Kgl. Danske Videnskabernes Selskab. Matematisk-fysiske Meddelelser.* VIII, 2, 1927.

Rasch, Georg (1928). Über die Nullstellen der unvillständigen Gammafunktionen $P(z, \rho)$. I. *Matematische Zeitschrift,* Vol. 29, 300−318.

Rasch, Georg (1930) *Om Matrixregning og dens Anvendelser paa Differens- og Differentialligninger.* København. Doctoral Thesis

Rasch, Georg (1931). Notes on the Gamma-function. *Annals of Mathematics, second series 32,* 592−599.

Rasch, G. (1952). Statistik. Nordisk Sommeruniversitet 1951. Aarsagsproblemet. København 1952. p. 244-251.

Rasch, G. (1955). Statistisk belysning af jorden og arbejdskraftens produktivitet. Virkelighed og beskrivelse. Nordisk Sommeruniversitet, Otnäs 1954. København 1955. p. 167-172.

Rasch, Georg (1959a). Processer og udvikling. Nordisk Sommeruniversitet 1958. Udviklingsproblemer. København 1959. p. 86-96.

Rasch, Georg (1959b). Om vurderingens indslag i erkendelsen. Nordisk Sommeruniversitet 1959. Erkendelse, Vurdering og Valg. Fire indledningsforelæsninger af Alf Ross, Georg Rasch, Mogens Blegvad og Mogens Pihl. Trykt som manuskript København 1959. p. 23-33.

Rasch, Georg (1960). *Probabilistic Models for Some Intelligence and Attainment Tests.* Copenhagen: The Danish Institute of Educational Research.

Rasch, Georg (1980). *Probabilistic models for some intelligence and attainment tests.* Copenhagen: DPI (Expanded edition of Rasch (1960). Chicago: The University of Chicago Press.

Wright, Benjamin (1980). *Foreword* Probabilistic models for some intelligence and attainment tests. Copenhagen: DPI (Expanded edition, 1980, Chicago: The University of Chicago Press. (pp. ix−xix).

Årbog (1944–1945). *Årbog for Københavns Universitet, Kommunitet og Den Polytekniske Læreanstalt (Danmarks Tekniske Højskole) indeholdende meddelelser for det akademiske år 1944-1945.* København.

Årbog (1958–1963). *Årbog for Københavns Universitet. Indeholdende meddelelser for de akademiske år 1958-1963.* København.

Georg Rasch's Growth Model

Lina Wøhlk Olsen[1]

Chapter 2

ABSTRACT This article explores Georg Rasch's work on growth.
Rasch started developing the Growth Model in 1940 and for some ten years to come his main interest was on it's applicability to account for the growth of various species, human babies included. In the beginning the Growth Model was deterministic, but at some point in time Rasch found a solution to the 'stochastic differential equation' underlying the model. Deriving this stochastic process seems to have been a special interest of Rasch's in the early 1950s. From the mid 1950s to the mid 1960s Rasch did not work much on growth, as he was deeply involved in the development of the Rasch models and the concept of specific objectivity. When Rasch started using the Growth Model again his main interest was how the Growth Model fitted inside the concept of specific objectivity.

1 Introduction

Though famous for the development of the Rasch models, Georg Rasch (1901-1980) worked with various statistical models. Working as a statistical consultant Rasch seldom got time to write down his ideas for publications. One of the ideas that Rasch never published is a model for analysing growth. This Chapter explores Rasch's work on growth. His work on the subject is believed to be of importance by 'his students', David Andrich and Peter Allerup.

As it is, little material on Rasch's Growth Model is public available. Publications on the subject narrows down to Rasch (1972) combined with a paper of C.S Rao (Rao, 1958). Therefore, much research went into finding sources.

Eight written sources were found. They cover a period in time of some 35 years. The oldest source is from 1940 and the most recent one is from the mid 1970s. The interesting thing is that Rasch throughout this period of time found new aspects of the Growth Model to concentrate upon. In the beginning his main concern was the model itself and it's applicability to describe the growth of various species. In the early 1950s he acquired an additional interest in modelling the random variations.

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This included deriving a solution to a 'stochastic differential equation'. In the 1960s and 1970s Rasch devoted himself to scientific investigations on the concept of specific objectivity, a concept that originated in the Rasch models. At this period in time Rasch found it especially interesting that the Growth Model fitted inside the concept of specific objectivity.

Since this paper is based on unpublished papers, most of which nobody remembers, Section 2 gives a description of these sources and where they were found. Section 3 gives a brief introduction to Rasch's model for analysing growth. Sections 4 through 7 explore how Rasch developed his ideas on growth, including how he changed his point of view upon what was most important. Finally, Appendix A is an empirical example of using Rasch's Growth Model.

2 Presentation of the sources

Eight written sources have been found on Rasch's work on growth. Amongst these, two are published, namely a paper written by C.S Rao, published in Biometrics (Rao, 1958) and a paper written by Rasch, published in the Danish journal, Nationaløkonomisk Tidsskrift (Rasch, 1972).

Four of the sources were traced down in Rasch's library, which is the name we shall use for the papers and books Rasch had in his home office. When Rasch at about his retirement in 1971 moved from his large villa in Holte to his weekend cottage on Læsø, his home library was arranged in some 25 old wooden beer boxes, and a friend of Rasch, Arne Jensen, brought them to the Danish Technical University. Rasch's library has remained at this place, almost untouched, until they were moved in 2001 to the University of Copenhagen, Institute of Economics. Rasch's library contains everything from books to stray notes and numerical computations. As it is, little order exists in these wooden beer boxes.

In his paper Rao mentions a series of lectures Rasch gave in Calcutta in 1951 (Rao, 1958). Much searching through Rasch's library has resulted in retrieving these notes, which we shall call *the Calcutta notes*. They consist of 235 pages, most of which are hand written. The Calcutta notes have the appearance of a rough draft. From time to time several pages are quite coherent, and a few are actually typewritten. But at other times, the pages mostly consist of stray notes. About half of the 235 pages are tables of data, numerical computations and graphs.

In Rasch's library, next to the Calcutta notes, was found yet a set of notes on

growth. These are in Danish, and we shall refer to them as *the Danish notes.* The Danish notes consist of 74 pages, of which the last 42 pages are graphs and numerical computations. Most of the first 32 pages are typewritten, with a lot of hand written corrections. The hand written corrections do not resemble Rasch's handwriting. This evidence, combined with the fact that some of the type written pages carries the header *Dr. Rasch. 2. lecture. The 6th of May 1952* seems to indicate that Rasch gave some lectures on growth upon his return from Calcutta, and that he worked jointly with other scientist to polish the theory. The Danish notes carry a higher level of mathematical statistics than the Calcutta notes, and it is therefore believed that Rasch's audience was other scientists with a mathematical or statistical background. Though The Danish Notes are quite readable they cannot be said to give a complete account of Rasch's work. As is the case with The Calcutta notes they are a mixture of fairly comprehensive text and rough notes.

Both the Calcutta notes and the Danish notes mention Steensberg data set on calves' growth, and Rasch's initial work on these data was found in his library. This work consists of a large number of plots and numerical computations and 15 typewritten pages, all of them connected to when Rasch in 1940 acted as critic at Steensberg defence of his doctoral thesis.

The paper Rasch (1954) was also found in Rasch's library, but as opposed to the other sources, Rasch (1954) appears at a list of Rasch's most substantial papers (c.f. Appendix B). Rasch (1954) is in English and typewritten. Several copies of the source exist in the library. Another difference between this source and the others is that the word growth does not appear. Actually, there is no evidence in Rasch (1954) that it has anything to do either the Calcutta notes or the Danish notes. There is, however, no doubt that the theory for stochastic differential equations in Rasch (1954) is a further development of the theory in the Calcutta notes and the Danish notes.

One paper was found at The Danish Educational University (DPU)[2] where Rasch's student, Allerup, has preserved the notes and books Rasch had at his office at DPU. This paper, which we shall refer too as the DPU paper is not dated, but it seems to be an earlier and longer version of Rasch's publication Rasch (1977). The chapter about growth in the DPU paper is not included in Rasch (1977).

Finally, in 1974 Rasch was in Perth, Western Australia. Here, he lectured on various topics, including, of course, the Rasch models. He also talked about The Growth

[2]Formerly, DPU was the Danish Institute of Educational Research (DPI).

Model. One of the listeners took notes, and Andrich has kept these notes. We shall refer to these notes as Rasch (1974).

Another type of sources on Rasch's work on growth is interviews. I would like to thank Peter Allerup, David Andrich, Niels-Erik Jensen and Jon Stene for telling me what they remember on Rasch's work on growth (Int. Allerup, 04.09.2002), (Int. Andrich, 06.02.2002), (Int. Niels-Erik Jensen, 27.06.2001), (Int. Stene. 08.10.2002). A brief description of these people's connection to Rasch is given in Appendix C. Furthermore, I would like to thank David Andrich for letting me use the interview he carried out with Rasch in 1979. In this interview, which we shall refer to as the Andrich-interview, Rasch mentioned his work on growth.

3 The Growth Model. Deterministic version

In this section Rasch's Growth Model will be introduced as Rasch did in the Calcutta notes.

When first suggesting the Growth Model in the Calcutta notes Rasch used growth of bacillus as an example. Let t_0 be the time it takes one bacillus to divide into two, and let $c(t)$ be the total number of bacillus at time t. Ignoring random variations $c(t)$ will increase as a function of t according to the following pattern:

t	number of cells $= c(t)$
0	1
$1t_0$	2
$2t_0$	4
$3t_0$	8
$4t_0$	16
$\vdots$	$\vdots$

This table implies that the equation

$$\log(c(t)) = log(2^t) = t\log(2) \tag{2.1}$$

is valid for $t = 0, t_0, 2t_0, 3t_0, ...$

Rasch argued that though the growth of a larger organism is a more involved process, the basic principle is the same; a certain percentage of the cells divide after a specific time period; and this time period will probably depend on time itself (The

Calcutta notes, p. 26). Now, let $y_\nu(t)$ denote the size of organism number ν measured one way or another, for instance a calf, it's size measured by weight. Let $\tau_\nu(t)$ be a transformation of time for organism number ν, and let α_ν and β_ν be unknown parameters specific to organism number ν. An equation expressing the same basic principle as (2.1), but taking into account that the growth rate may vary from time to time, is:

$$log(y_\nu(t)) = \alpha_\nu + \beta_\nu \tau_\nu(t). \tag{2.2}$$

Of course, it is always possible to find a transformation of time, $\tau_\nu(t)$, such that (2.2) applies to each single organism. Therefore, Rasch's question was whether similar types of organisms would grow in similar ways (The Calcutta notes, p.26):

> *Now we may hazard the question whether the way in which a time interval counts at different ages is something specific for the type of organism considered. If so, we should, by measuring the age in a particular way, get a uniform description of the growth curves for all organisms of the type considered.*

That is, for all organism the τ_ν functions must be the same. Or equivalently, the deterministic model

$$log(y_\nu(t)) = \alpha_\nu + \beta_\nu \tau(t) \tag{2.3}$$

must apply to all organisms $\nu = 1, ..., n$. For further references we shall refer to (2.3) as *The Growth Model*.

According to Andrich, Rasch referred to τ as *the metameter* in the 1970s (Int. Andrich, 06.02.2002). Since this was the name Rasch decided on, we shall use it. However, in the Calcutta notes Rasch called the metameter for *the age transforming function*. It is this function that makes Rasch's theory so interesting.

As Rasch pointed out a metameter only exists if the n organisms growth in similar ways. A limitation on the time interval where the model is expected to apply is also required. For instance, Rasch found that $log(t + 2)$ could be used as metameter for some 170 calves; but only when the calves were older than half a month and younger than 20.5 months. To postulate that $\tau(t) = log(t + 2)$ could be used, for instance, until the calves were 70 months old would result in ridiculously large calves, hypothetical speaking that is. In the following The Growth Model is considered for $t \in]\check{t}, \tilde{t}]$.

The Growth Model may also be expressed as a differential equation. This formulation was very important in Rasch's work and an the Calcutta notes he even argued for the Growth Model in the shape of a differential equation (The Calcutta notes, p. 200):

> *The growth is an extremely involved process.* [...] *But* <u>*one*</u> *point seems reasonable all the same: The rate of the growth is within a relatively stable period proportional to the number of active cells, which its form is proportional to, say, the total weight of the organism.*

Following from this reasoning Rasch proposed the model

$$y'_\nu(t) = \beta_\nu \lambda(t) y_\nu(t) \tag{2.4}$$

We shall refer to this model by the Differential Equation of Growth.

Now, the Differential Equation of Growth can be obtained from the Growth Model by differentiating:

$$\frac{d\log(y_\nu(t))}{dt} = \frac{y'_\nu(t)}{y_\nu(t)} = \beta_\nu \tau'(t)$$

This implies that the metameter, $\tau(t)$, can be obtained from the the proportionality factor, $\lambda(t)$, and vice verse:

$$\lambda(t) = \tau'(t) \quad \tau(t) = \int \lambda(t)$$

Rasch's Growth Model as it appears in equation (2.3) is deterministic. In the Calcutta notes and in the Danish notes Rasch developed theory to take random variation into account. This will be described in Section 5.

4 Rasch initial development of the Growth Model

This section will investigate how Rasch developed The Growth Model and the Differential Equation of Growth.[3] In this respect it is noteworthy that Rasch introduced the Calcutta notes by commenting on this:

> *The studies put down in the present publication are not the result of systematic investigations on growth and variability, much less do they pretend to form a monograph on the subject. They are in fact nothing more than some rather casual observations during 20 years work in the biometric field*

[3] Section 4 is based on on the Calcutta notes and the Andrich-Interview.

which never left me much time for systematic research on my own behalf. Acknowledgements to all who have contributed to the work by generously placing data at my disposal.

In 1936 I had the unique opportunity of making extensive studies of the then existing literature on the qualitative aspect of growth under the guidance of Julian Huxley. Since then, I regret to say, I have had no opportunity of following up systematically the insight thus gained. From conversations with biologists and biometricians as well as from my casual contact with the more recent literature on growth I have, however, gathered that the approach here presented is essentially new. Therefore the presentation, for what it is worth, and my apologies in advance for any omissions of quotations.

In short, the events that led Rasch to encounter problems of growth in the mid 1930s may be summarized as follows:[4] In his youth Rasch hoped for a position in academia as a mathematician, but as it turned out there was no work in mathematics. Not wanting to quit science completely, Rasch earned to make a living as a statistical consultant, in the beginning at the Hygienic institute[5] and the State Serum Institute[6].

Because Rasch had the backing of influential circles, amongst others his former professor of mathematics at the University of Copenhagen, Niels Erik Nørlund, and the then director of the State Serum Institute, Thorvald Madsen, Rasch managed to get a Rockefeller scholarship for a years study in London by R.A Fisher, undoubtedly the most famous and influential statistician who ever lived. Consequently, Rasch studied in London in 1935 and 1936, when he was 34 years old. Now, in the Andrich-Interview Rasch tells about working with growth already while studying in London:

I had brought with me some data concerning the growth problems of school children which I had obtained from the Hygienic Institute and I worked a bit on them, trying to use Fisher's methodology. That brought me into connection with Julian Huxley, an extremely well known biologist. He had taken up something that interested me very much. He had written a book by the title "Problems of Relative Growth"[7]. What he did there was actually for particular kinds of animals. He got many of his data from records

[4]The information about Rasch's youth is based on Andersen(1980).
[5]In Danish: Hygiejnisk Institut.
[6]In Danish: Statens Serum Institut.
[7](Huxley, 1932)

> *of various animals shot by people for sport or whatever they do it for. Record was kept about the great animals that they shot. He plotted then the weight of particular organs, say, against the rest of the animal. Doing that logarithmically he found, what surprised him and many others what were very close to straight lines. So the weight of an organ is proportional to the weight of the rest of the animal in some power* [...] *Then I showed him some data I had found.*

So, Rasch's first analysis of growth was an analysis of children's growth, and an analysis of the data he showed Huxley. These later data were about growth of crabs. No evidence of either analysis has been found in Rasch's library, implying that the only information about their nature is Rasch's brief description of his work in the Andrich-interview. From this source it is evident that Rasch's analysis of the crab data set was much more important than his analysis of children's growth. Rasch's statement in the Andrich-interview is not detailed enough to get a complete picture of what he actually did, but it is evident that he transformed the data logarithmically, and then plotted them against a proxy for age. The resulting plots resulted in straight lines.

As we shall see, in later works on growth, Rasch plotted the logarithm of the size of an animal against a function of age. And, by doing so, he obtained straight lines. This seems to suggests that Rasch's analysis of crabs was an early version of his later Growth Model. Too little information is available to further demonstrate this.

It seems that Rasch found this encounter with Huxley important, the point being that Rasch's analysis aimed at an analysis of the *individual*, whereas Huxley analysed populations. In the Andrich-Interview Rasch said:

> *Well, I wouldn't call that the highlight of my life, but is something that shows that fairly early, I got around the problem of dealing with individuals. I had tried to do that for the growth of children already before I came to London. But meeting Julian Huxley showed me that this was really an important line of my research. I continued to stick, as far as I could, to the study of individuals ever since. It meant quite a lot to me to realize the meaning and importance of dealing with individuals and not with demography.*

Judging by Rasch's statement in the Calcutta notes his next important encounter with the analysis of growth took place in 1940 when Steensberg defended his doctoral thesis. The thesis was a large-scale growth experiment with a particular breed

of calves, namely 'Red Danish Milk Breed'. Three groups of calves consisting of respectively 58, 62 and 51 calves were weighted and measured once a month starting from the age of a half month and continuing until the calves were 20.5 months old. Each of the three groups was given different supplementary fodder.

Trying to make sense of all these numbers Steensberg's thesis was on the borderline to statistical biology. Rasch, in his capacity of a statistician, was chosen to act as critic. For further references we shall call Rasch's comments on Steensberg's thesis for The Critique. The Critique consists of 15 typewritten pages.

After having criticized Steensberg's analysis very thoroughly and pointed out several flaws and mistakes, Rasch suggested what Steensberg might have done instead:[8]

> *It is well known that simple organisms like for instance coli bacillus under optimal growth conditions propagate according to a quite simple multiplicative law: in the course of a certain time unit, one bacillus becomes 2, in the course of the next unit, 2 becomes 4 etc. And this way it will continue; as long as the environment is not poisoned and as long as there is nourishment enough for all of the bacilli. Every increment is, to be sure, due to cell divisions; for higher animals of course of a much more involved nature than for coli bacillus, but in principle it is the same. But, especially when higher animals are involved, as for instance calves, the "age", with all what comes with it in alterations of the inner and outer environment, plays a decisive role. But then it is tempting to think of the underlying growth principle in much the same manner, except that the calendar times at some stages are much more valuable for the growth than at other times - that is, downright to sustain the simple elementary growth law, the relative constant increase per unit of time, as long as the time is expressed in the physiologically adequate unit of time. In this respect the conclusions of my work with your data has been very encouraging. They may be interpreted as - within the limited time interval from half a month to 20.5 months - the mentioned simple growth law actually does apply to each of your calves with respect to all 6 quantities you have measured and this is irrespective of how they have been fed; as long as the physiological unit of time applied is the logarithm to age, measured chronologically, plus 2 months.*

[8]The quote is translated from Danish. The Danish text can be found in Appendix C

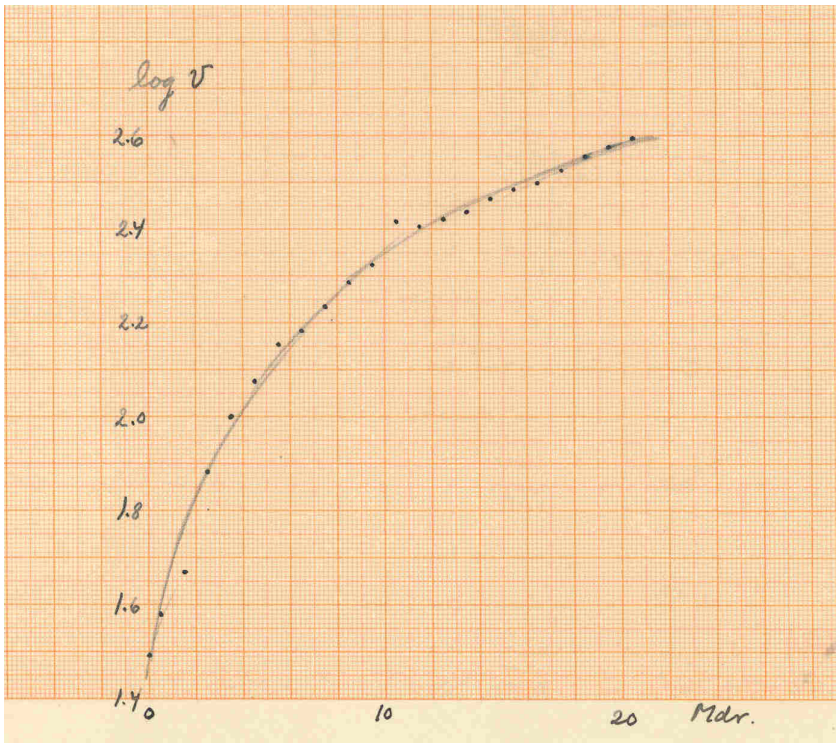


FIGURE 1. *$log(weight_\nu(t))$ plotted against time measured in months for calf number A54.*

An extensive amount of plots and numerical computations found next to The Critique in Rasch's library verify that Rasch did an extensive analysis of Steensberg's data. Figure 1 and Figure 2 are examples of such plots.

No model was written down algebraically in The Critique. Neither has such been found amongst the plots of Steensberg's data. There is, however, strong evidence that the model underlying Rasch's work in the Critique is identical to the Differential Equation of Growth as stated in Equation (2.4). This is seen as follows:

If the Growth Model (2.3) is perceived as a function of τ, that is

$$log(y_\nu(\tau)) = \alpha_\nu + \beta_\nu \tau,$$

we have that

$$\frac{d}{d\tau} log(y_\nu(\tau)) = \frac{y'_\nu(\tau)}{y_\nu(\tau)} = \beta_\nu. \tag{2.5}$$

But equation (5.16) states that the relative increase per unit of time is constant, as long as the time is expressed in a physiologically adequate unit of time, just as Rasch wrote in the Critique: " - *that is, downright to sustain the simple elementary growth law, the relative constant increase per unit of time, as long as the time is expressed in the physiologically adequate unit of time.*". This is reinforced by the

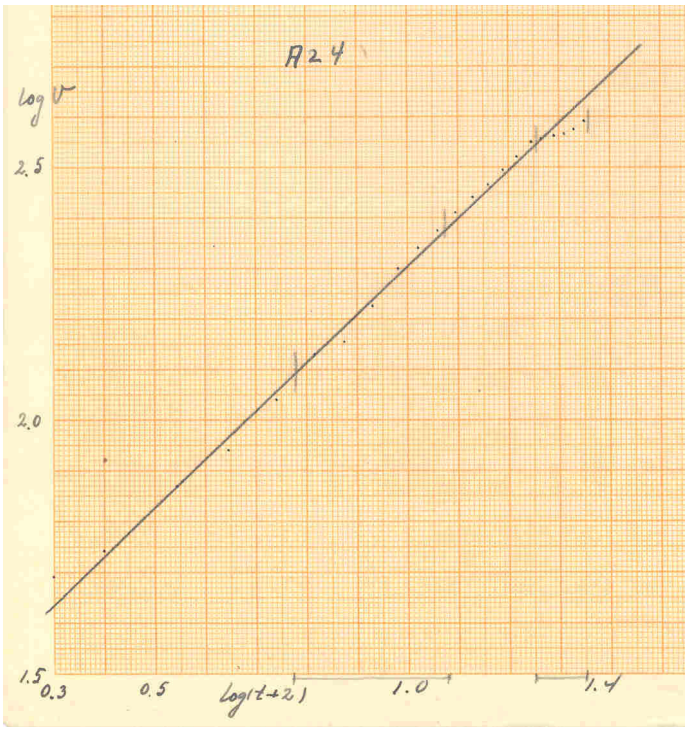


FIGURE 2. *$log(weight_\nu(t))$ plotted against $\tau(t) = \log(t+2)$ for calf number A24.*

similarity between the bacilli example in the Critique and the one Rasch gave in the Calcutta notes (c.f. Page 60). Another argument that Rasch was using model (2.3) in the Critique is to look at plots such as Figure 2. To check whether a plot of $(x, y) = (log(t+2), log(weight(t)))$ yields straight lines for all calves considered is to check the fit of model (2.3). Therefore, there is no doubt that Rasch was using The Growth Model in 1940, though perhaps primarily formulated as the Differential Equation of Growth.

After acting as critic in 1940 at Steensberg's defence, Rasch worked with other data sets on growth. He analysed growth of bacillus, guinea pigs, mice, rats and human babies. We shall not go into detail with these growth experiments. The important thing is not the specific analyses, but the fact that Rasch from time to time analysed growth. However, one specific encounter with growth forced Rasch to make progress. In the early 1940s when analysing a data set, Rasch found the mathematical form of the metameter by 'guessing'. For instance, when he analysed Steenbjerg's data, Rasch 'guessed' that the functional form of the metameter was $\tau(t) = \log(t+2)$. At some point in time he encountered a data set where he was not able to find the functional form of the metameter. Rasch wrote about this in the Calcutta notes (p.

36-37):

> *During the years Lindholm carried out several series of experiments on the effect of various A vitamin preparations on the growth rate of rats.* [...] *In these experiments I did not succeed in guessing a simple age transforming function which could play the same role for the rats as did* $\log(t+2)$ *for the calves. Thus the question arose whether such a function existed or not. The answer was given by simply constructing a function which must be an age transformer in case such a function exists, the construction being followed by of graphical control of whether this function actually possessed the property wanted.*

The way Rasch constructed the metameter was as follows: Assume that n organisms can be described by model (2.3), and let $\bar{x}$ denote the average of $x_1, ...x_n$. Averaging in (2.3) at time t we have that

$$\overline{\log(y(t))} = \bar{\alpha} + \bar{\beta}\tau(t).$$

An estimate of the metameter can therefore be obtained as

$$\tau(t) = \frac{\overline{\log(y(t))} - \bar{\alpha}}{\bar{\beta}}$$

which implies that

$$\log(y_\nu(t)) = \alpha'_\nu + \beta'_\nu \overline{\log(y(t)}. \tag{2.6}$$

That is, *if* a metameter, τ, exists the average, $\overline{\log(y(t))}$, can be used as an estimate. Likewise can the sum, $\sum_{\nu=1}^{n} \log_\nu(y(t)$.

It has not been possible to find out exactly when Rasch developed the method of calculating the metameter. Some data in the Calcutta notes concerns growth of rats, and this experiment seems to have been carried out in 1944 and 1945 (The Calcutta notes, p. 31-35). Since there is no guarantee that this is the correct data set or, given that it is, that Rasch started working on the data immediately after the experiment had been carried out this gives no unambiguous information. The only thing that is certain is that Rasch found out how to 'estimate' the metameter after 1940 but before 1951, which was the year Rasch was in Calcutta.

Now, in 1951 Rasch was invited to give a series of lectures in Calcutta at the 1951 meeting of the International Statistical Institute (I.S.I.). According to Rasch this was

the first time he talked about the calves outside Denmark (The Andrich-Interview, p. 49). In the Andrich-interview Rasch said about his lectures in Calcutta:

> *I had a collection of some different kind of animals and among them human babies. For all of them I found what I now call a Growth Mode and could estimate a growth rate. Well I had that, and of course, some other bits of mathematical statistics of my own invention.*

These 'bits of mathematical statistics of his own invention' especially had to do with finding transformations of random variables such that they became normally distributed. Rasch also developed a theory for dealing with random variations in the Growth Model. But as it is, these 'bits' only play a minor role in the Calcutta notes. There is no doubt that the Growth Model and the corresponding Differential Equation of Growth were emphasized the most. And it is evident that Rasch was especially interested in convincing his audience of the applicability of the Growth Model to various species. The following two quotes substantiates this (The Calcutta notes, p. 45):

> *In conclusion I may mention that a preliminary investigation tends to show that man is no exception to the rule found. Human babies.*

and (The Calcutta notes, p. 201):

> *The empirical law tells that*
>
> $$\frac{d\log y_\nu(t)}{dt} = \beta_\nu \lambda(t) x_\nu(t)$$

Round about half of the 235 pages which constitutes the Calcutta Notes are plots, tables of raw data and calculations. This further indicates that Rasch aimed at demonstrating the applicability of the Growth Model to various species; The plots and raw data stands as empirical evidence.

Now, one among Rasch's audience was convinced of the applicability of the Growth Model. Rasch said in the Andrich-interview:

> *Rao was one of my students there and he was quite fascinated by it. He actually wrote in Biometrics some years later, a paper on the basis of my ideas, which I hadn't published by then.*

C.R. Rao is a well-known statistician. The article Rao published in Biometrics was titled *Some Statistical Methods for Comparison of Growth Curves* (Rao, 1958). In this

article Rao used Rasch's Growth Model, though without the logarithmic transformation. He estimated the metameter as in equation (2.6), and acknowledged Rasch for this method (Rao, 1958, p. 3). It is outside the scope of this paper to enter a discussion on the impact that Rao had. Neither will it be commented upon that Rao and not Rasch is credited for the idea of estimating the metameter from data (Izenman and Williams, 1989). The fact that Rao published a paper using Rasch's idea of including a metameter in the model must, however, be taken as indication that Rao, and the referees of Biometrics at that time, found Rasch's ideas on growth noteworthy.

5 Growth as a stochastic process

The year after Rasch came home from Calcutta, that is, in 1952, he gave a seminar on the Growth Model. His notes to this seminar are *the Danish notes*. As opposed to the Calcutta notes the main objective of the Danish notes was to describe the random variations when dealing with the Growth Model.[9]

In 1954 Rasch gave a lecture at the university of Copenhagen, Institute of Statistics (Rasch, 1954). Seemingly, the paper he then presented had noting to do with the Growth Model. However, the differential equation considered is seen to include the Growth Model as a special case.

We therefore have three sources where Rasch worked on a theory for random variation in connection with a differential equation such as the Differential Equation of Growth (2.4). Apparently, Rasch had not entirely worked out the theory in the Calcutta notes, neither in the Danish notes. But both sources seem to converge towards the contents of Rasch (1954), with an increasing level of rigour and mathematical sophistication. We shall limit our attention to this convergence, omitting the other more or less finished suggestions that can be found in the Calcutta notes and the Danish notes.

The Calcutta notes includes a chapter or a section titled *Growth as a stochastic process* that takes random variations into account when dealing with the Growth Model. As the chapter begins at page 200 out of a total of 235 pages, it seems that only a minor part of Rasch's lectures were concerned with this aspect. The chapter has the appearance of being a rough draft. Some pages are quite coherent, but from

[9]Section 5 is based on the Calcutta notes, the Danish notes and Rasch (1954).

time to time it seems as if pages are missing.

From the coherent part of the Calcutta notes, it is seen that Rasch rewrote the Growth Model (2.3) as (the subscript, ν, has been omitted to ease the notation):

$$\log(y(t)) = \quad \alpha + \beta\tau(t) + u(t),$$

where $u(t)$ represent some sort of random variation. The purpose of this expansion was to estimate the parameters α and β (The Calcutta notes, p. 202). In the Calcutta notes Rasch gave at least two suggestions as how to model the random variation. Only one of these approaches can be found in latter sources, and it must be assumed that Rasch did not find the abandoned approach satisfactory. Now, in the approach that Rasch continued to work alongside he suggested that "*we may think of good solid disturbances, actually occurring now and again, but at irregular intervals* (The Calcutta notes, p. 205)." Rasch assumed that the equation

$$\frac{d}{d\tau} log(y(\tau)) = \beta.$$

was valid most of the time. But, at time $t_1^{'}, ..., t_k^{'} \in]0, \tilde{t}]$, disturbances or 'jumps' of size respectively $\varepsilon_1, ..., \varepsilon_k$ occurred:

$$\log(y(t_j + 0)) - \log(y(t_j - 0)) = \varepsilon_j.$$

Consequently, within the limited interval of time, $]0, \tilde{t}]$, the total logarithmic gain of the process was:

$$\log(y(\tilde{t})) - \log(y(0)) = \alpha + \beta\tau(\tilde{t}) + \sum_{j=1}^{k} \varepsilon_j$$

Both in the Calcutta notes and in the Danish notes Rasch considered the case where the ε_j's were assumed independent, normal distributed[10] with mean zero and variance σ^2. The number of 'jumps', k, was assumed to follow a Poisson distribution with intensity λ, but no assumptions were made as to how the k error terms were distributed amongst the times at observations $t_1, ..., t_n$.

It may be noted that the stochastic process Rasch defined is a special case of a *Piecewise Deterministic Markov Process*, which was introduced by Davis in 1984 (Davis, 1984). Rasch's objective of introducing such a process was not to study it for it's own sake. He was more interested in considering case that correspond to a continues flow of disturbances, that is, where the number of 'jumps', k, approaches

[10]Later in life Rasch was known to be quite annoyed with the normal distribution, or rather how it was used and especially how it was misused (Int. Allerup, 26.02.1998). As is evident here, Rasch used the normal distribution himself.

infinity, which is the case when $\lambda \to \infty$. That is, instead of writing down a stochastic differential equation as we would today, for instance,

$$dY(t) = \beta\tau'(t)Y(t)dt + \sigma\sqrt{\tau'(t)}Y(t)dZ(t),$$

where $Z(t)$ is a Brownian motion, Rasch introduced 'good solid jumps' and then approximated a continuous flow of disturbance by considering the limiting case, $\lambda \to \infty$.

In the Calcutta notes Rasch stated for the limiting case, $\lambda \to \infty$, that the random variables,

$$\log(y(t_j)) - \log(y(t_{j-1})) - \beta(\tau(t_j) - \tau(t_{j-1})), \tag{2.7}$$

$j = 1, ..., n$, are independent, normal distributed with mean zero and variance $\sigma^2(\tau(t_j) - \tau(t_{j-1}))$. The Calcutta notes do not contain a proof of this statement, but the Danish notes do. It has not been possible to find out whether the pages containing the proof have gone missing over the years, or if Rasch simply hadn't worked out the theory in 1951.

Having found the limiting distribution of (2.7) Rasch easily found the maximum likelihood estimate of β as (The Calcutta notes, p. 218):[11]

$$\hat{\beta} = \frac{\log(y(t_n)) - \log(y(t_o))}{\tau(t_n) - \tau(t_0)} \tag{2.8}$$

In this case α is simply estimated by the starting value (The Calcutta notes, p. 219).

In the Calcutta notes Rasch mentioned that the estimate of the β's depends strongly on the assumptions made; especially the endpoints are critical. He therefore gave some guidelines as how to control them. It is only to be expected that Rasch would emphasize the importance of checking out any statistical assumptions made (see for instance Andersen (1995)). But, this evidence also tells that Rasch had his mind on applications when he lectured in Calcutta.

Turning towards the Danish notes, the major difference between the Calcutta notes and the Danish notes is that whereas Rasch in the Calcutta notes emphasized the applicability of the Growth Model to various species, and only considered the problem of dealing with the random variations in the last 35 pages, in the Danish notes, Rasch went straight to the problem of constructing a stochastic process, which is a solution to the Differential Equation of Growth. Basically, he went through the

[11] The model correspond to a regression model with weights $1/(\tau(t_i) - \tau(t_{i-1}))$ and no intercept, but the estimate reduces to (2.8).

same theory as in the Calcutta notes, but on a higher mathematical level. He also went through theory that was not dealt with in the Calcutta notes. Especially the problem of how to derive the limiting distribution of the random variables of Equation (2.7) was analysed thoroughly. And the Danish notes prove that the limiting distribution is in fact a normal distribution, and that the successive differences, $\log(y(t_i)) - \log(y(t_{i-1}))$ are independent. In the Danish notes Rasch also calculated estimates of α and β under the assumption that the 'jumps', $\varepsilon_1, ..., \varepsilon_k$, followed a gamma distribution. These derivations are very incoherent, and no further comments will be made. The main issue here is not the theory in itself, but the fact that when comparing the Danish notes with the Calcutta notes, it is evident that Rasch's focus had changed such that his main interest was to find a solution to a stochastic differential equation. In this respect it may be noted that Rasch at no point in time writes down a stochastic differential equation as we would today. He 'simply' models the errors as described earlier, and then arrives at something similar to a continuous flow of disturbance by considering the case $\lambda \to \infty$.

It seems that Rasch in the early 1950's became so fascinated of constructing a stochastic process that he studied it for it's own sake. The unpublished paper Rasch (1954) titled *Contributions to the Theory of Stochastic Differential Equations* substantiates this. Rasch talked about this paper at an informal seminar at the University of Copenhagen in 1954. In 1956 he lectured about it in Uppsala at the European Session of Econometric Society.

Rasch gave no references whatsoever to problems of growth in Rasch (1954). Instead the contribution of the paper was suggested to be a solution to a linear stochastic differential equation often used in econometrics. The differential equation Rasch considered in Rasch (1954) was

$$x'(t) = x(t)\alpha(t) + \phi(t), \tag{2.9}$$

where $x(t)$ and $\phi(t)$ are p dimensional vectors, while $\alpha(t)$ is a $p \times p$ matrix. The Growth Model, given by the differential equation (2.4), can be obtained from equation (2.9) by considering the one dimensional case $p = 1$ and letting $\phi(t)$ identically equal zero.

The way that Rasch modelled the disturbance in Rasch (1954) was quite similar to the approach taken in the Calcutta notes and in the Danish notes. He assumed

that the deterministic process

$$x'(t) = x(t)\alpha(t) + \phi(t)$$

proceeded almost always. At certain time points, say $t_1' < t_2' < t_3' < ...$ the process receives a series of independent random impulses $\varepsilon_1, \varepsilon_2, \varepsilon_3, ...$:

$$x(t_j + 0) - x(t_j - 0)) = \varepsilon_j. \tag{2.10}$$

It is further assumed that the time points $t_1', t_2', t_3', ...$ vary at random from one realization of the process to another, and that the disturbance free intervals are independent. The probability of a disturbance occurring in any infinitesimal time interval $(t, t + dt)$ is λdt, where λ is assumed to be constant, just as before. In Rasch (1954) Rasch also considered the limiting case, $\lambda \to \infty$, to be of special interest. He found that if the $\varepsilon_1, \varepsilon_2, \varepsilon_3, ...$ was normal distributed then the limiting distribution was normal. However, as opposed to the two earlier papers on growth, Rasch was not interested in finding estimates in Rasch (1954).

The paper Rasch (1954) is by no means easy reading, and Rasch's handling of the matrix algebra brings to mind that he wrote a doctoral thesis about the use of matrix algebra on differential and difference equations (Rasch, 1930). This, together with the lack of interest in estimating the parameters and the lack of examples clearly distance Rasch(1954) from the Calcutta notes and the Danish notes. It has, however, not been possible to find out if Rasch wrote the paper Rasch (1954) of pure interest or if he had other things in mind, as for instance documentation of his scientific work should a chair in statistics come within reach (c.f. Chapter 1 and Chapter 4 of this thesis).

6 Work with growth after 1954

In the mid 1950s Rasch became involved in statistical analyses of intelligence tests and psychological measurement (Andersen and Olsen, 2001). He did an increasingly amount of consulting within these areas. This work resulted in the models, which bear his name, namely the Rasch models, but a side effect was that Rasch had little use for his Growth Model and differential equations for some years to come. The first time known where Rasch used his Growth Model again was in the late 1960s. At this point in time Rasch was professor of Statistics at the University of Copenhagen, faculty of Social Science, and had been so since 1962. The students attending his courses were economists and sociologists. The Rasch models fitted nicely to the problems

the sociologists were facing, but they had little application in economics. There are indications that Rasch began to analyse trends in economic because of a more or less direct critique of his focus on problems faced in sociology (Int. Niels-Erik Jensen, 02.11.2001), but it has not been possible to further document this. It is, however, a fact that Rasch used his Growth Model to analyse trend of wages and salaries in the lecture he gave when he retired as professor (Rasch, 1972), and Allerup remembers that Rasch analysed trend of prices on various consumption goods (Int. Allerup. 04.09.2002). Actually, several people remember Rasch using the Growth Model on economic data (Int. Andrich, 04.02.2002), (Int. Niels-Erik Jensen, 02.11.2001), (Int. Stene, 08.10.2002). For instance, Andrich tells that Rasch later would say that it did not strike him as surprising that growth in economics from times to times could be described by the same model as the one he had developed for the growth of animals, the point being that economic growth usually is perceived as a percentage of some existing magnitude, just like the growth of any animal.

In 1974 Rasch was invited to Perth, Western Australia. On this occasion he gave lectures on various topics, including of course the Rasch models. But he also talked about the Growth Model. The source Rasch (1974) is the notes taken by one of the listeners. From these notes it appears that Rasch used the Growth Model to analyse growth of pigs. According to Andrich, Rasch got these data from a quite advanced pig farm that was placed on the Danish island, Læsø, where Rasch had a weekend cottage (Int. Andrich, 06.02.2002). The interesting thing is that it appears from Rasch (1974) that Rasch presented the Growth Model in basically the same way as he did in the Calcutta notes when he was in Australia, that is, Rasch's emphasis was on the applications of the model, in this case in particular it's application to analyse the growth of pigs. And though Rasch also gave a short introduction to growth as a stochastic process his purpose was mainly to estimate the parameters of the Growth Model. The focus of the source Rasch (1974) is therefore similar to that of the Calcutta notes, which seems to suggests that even though Rasch from time to time at a theoretical level became fascinated by the problem of describing the disturbance, in practice, Rasch would use his Growth Model basically in the same way as he did in 1951.

7 The Growth Model and specific objectivity

In connection with the development of the Rasch models, Rasch developed the concept of *specific objectivity*.[12] This concept remained his main interest in sciences ever after. The interesting thing is that when Rasch about 1970 began to use the Growth Model again he placed it within the frame of specific objectivity. The two sources, Rasch (1972) and the DPU paper analyse the connection. The complicated theory Rasch developed to deal with the random variation was, on the other hand, completely left out of the discussion. A short introduction to specific objectivity will prelude the analysis of the connection between Rasch's Growth Model and specific objectivity.

When dealing with specific objectivity Rasch dealt with two different situations, namely a deterministic and a probabilistic. Fundamentally, the demands to specific objectivity were the same in the two situations, but in practice there were deviations. Since Rasch did not take random variation into account when he framed the Growth Model within the concept of specific objectivity we shall limit our attention to the deterministic case.[13]

TABLE 2.1. The agents, a_ν is confronted with the objects, o_i, resulting in a response, $r_{\nu i}$.

ν / i	o_1	$\cdots$	o_k
a_1	r_{11}	$\cdots$	r_{1k}
$\cdots$	$\cdots$	$\cdots$	$\cdots$
a_n	r_{n1}	$\cdots$	r_{nk}

Rasch's objective with specific objectivity was to lay down rules for making proper comparisons. He considered the case of a two dimensional table such as Table 2.1: Each time an agent, a_ν, is confronted with an object, o_i, a response is observed:

$$r_{\nu i} = \mu(a_\nu, o_i), \quad \nu = 1, ..., n. \quad i = 1, ..., k.$$

The agents, the objects and the responses are assumed to be completely described by real, one dimensional parameters, respectively the a_ν's, the o_i's and the $r_{\nu i}$'s.

[12] Section 7 is based on Rasch (1972) and the DPU paper.

[13] The following account is based on (Rasch, 1972, p. 168-172). It must be pointed out that Rasch's never quitted developing specific objectivity. Consequently, it is possible to find sources where Rasch define the concept somewhat differently. The account given here should suffice to illustrate the connection to his Growth Model.

Rasch's demands to specific objectivity were that comparisons of two agents, based on their responses to the objects, should be independent of which particular objects they had responded to, and also independent of responses of other agents. Likewise, comparisons of two objects should be independent of which agents that had responded to them, and of other objects present. If for examples the objects were questions in a questionnaire and the agents were students the demands were that it should be possible to compare the level of the students independently of which specific questions they had responded to and also independently of other students. Likewise, it should be possible to compare the level of the questions independently of which particular students had responded to them, and also independent of the other questions in the questionnaire.

Rasch's initial motive for introducing the Growth Model was in a way similar to specific objectivity. The data were lined out in a two way table such as Table 2.1, categorized by the individual calves and the time of measurements, and his goal was to compare Steensberg's calves *independently of time*. Rasch for instance wrote in the Critique:[14]

> *The fact that you at page 63 draw some curves of the progress of these increments does not remedy the fundamental uncertainty towards the circumstance that these numbers have to cohere one way or other - that is, obviously we cannot evaluate the disparity between the groups of calves at two different point in times without getting involved in the most terrible statistical dilemmas.*

And after having introduced the Growth Model as the solution to Steensberg's problem Rasch wrote:[15]

> *If you are in possession of this principle, well, then almost everything will come along. For instance, the growth curve for each individual animal is then, except for more or less random variations, determined by 2 constants.* [...] *With this tool in hand it is possible to occupy oneself with literally speaking each and every of the investigations which you have wanted to perform but which you have lacked the tools to do.*

Since the Rasch's initial motive for introducing the Growth Model was similar to what is required of specific objectivity it is not surprising that the Growth Models

[14]The text has been translated from Danish. The original text is found in Appendix C.
[15]The text has been translated from Danish. See e.g. Appendix C for the Danish text.

fits into the concepts. We shall state Rasch's main theorem for specific objectivity to elaborate on the connection:

> Main Theorem 1. (Rasch, 1972, p. 171)
> *A necessary and sufficient condition for making specifically objective comparisons of the objects and of the agents is that the responses, say $r_{\nu i}$, are latent additive functions of the agent parameters, a_ν, and the object parameters, o_i, $\nu = 1, ..., n$, $i = 1, ..., k$. This means that there must exist three real (and differentiable) functions f, g and h such that*
>
> $$f(r_{\nu i}) = h(a_\nu) + g(o_i).$$

As an example on how to make a specific objective comparison, it may be noted that if the responses are latent additive functions we have that

$$f(r_{\nu_1 i}) - f(r_{\nu_2 i}) = h(a_{\nu_1}) - h(a_{\nu_1}).$$

But this implies that we can compare the two agents, a_{ν_1} and a_{ν_2}, independently of all objects and also independently of other agents based on their responses to any object, o_i. And this is the very definition of making specifically objective comparisons of a_{ν_1} and a_{ν_2}.

Returning now to Rasch's Growth Model, and how he framed it within the concept of specific objectivity, the first written source is the lecture Rasch gave when he retired as professor at the faculty of Social sciences, Rasch (1972). In this paper Rasch mostly talked about specific objectivity, but he also gave empirical examples, two of them connected to the Growth Model.

The fist of these examples will suffice as an illustration of the connection between specific objectivity and the Growth Model. The example is about hourly wages in nine different industries. For each industry, ν, the average wage per hour, say $y_\nu(t)$, is observed each year, t, $t \in \{1953, 1954, ..., 1969\}$. Rasch proposed the model

$$y'_\nu(t) = \beta_\nu \tau'(t) y_\nu(t) \tag{2.11}$$

(of course making sure that it fitted the data). Model 2.11 is recognized as the Differential Equation of Growth.

Since (2.11) integrates to

$$log(y_\nu(t)) = \alpha_\nu + \beta_\nu \tau(t), \tag{2.12}$$

and since the function $\alpha_\nu + \beta_\nu\tau(t)$ is not latently additive, Main Theorem 1 states that it is *not* possible to make specifically objective comparisons based on the observed hourly wages, $r(\nu, t) = y_\nu(t)$. As a matter of fact, the parameter that describes the industries, namely (α_ν, β_ν), is not even one dimensional, which means that the model (2.12) falls outside the framework for specific objectivity that Rasch proposed.

Rasch solved this problem by rewriting (2.11) as a process, namely

$$\frac{y'_\nu(t)}{y_\nu(t)} = \frac{d}{dt} log(y_\nu(t)) = \beta_\nu\tau'(t). \tag{2.13}$$

The function, (2.13), is latent additive and it is therefore possible to make specific objective comparisons of the β_ν's and of the $\tau'(t)$'s. The parameters, $\alpha_1, ..., \alpha_9$, on the other hand, are outside the framework of specific objectivity.

The disadvantage of making specific objective comparisons by using as response function $r(\nu, t) = \frac{d}{dt} log(y_\nu(t))$ is that it is not immediately observable. In the DPU paper, Rasch offered another choice which remedies this flaw, namely

$$\begin{gathered} log(y_\nu(t_{i+1})) - log(y_\nu(t_i)) = \\ \alpha_\nu + \beta_\nu\tau(t_{i+1}) - (\alpha_\nu + \beta_\nu\tau(t_i) = \\ \beta_\nu(\tau(t_{i+1}) - \tau(t_i)). \end{gathered} \tag{2.14}$$

In both equations, (2.13) and (2.14), the mean by which specific objectivity is obtained is, basically, to consider relative changes in the logarithmic values.

Judging by the two sources, Rasch (1972) and the DPU paper it appears that Rasch was much more interested in specific objectivity than the Growth Model. Actually, the Growth Model was merely introduced as a special case of specific objectivity, whereas none of the impressive theory Rasch developed for dealing with the random variations was considered. This is in agreement with the fact that Rasch about this time devoted most of his time to further developing specific objectivity. It was his main interest in life, actually to the point where he was recognized as a missionary for the concept (Int. Allerup, 26.02.1998). On the other hand, the lectures Rasch gave in Australia in 1974 documents that Rasch in *practice* still used the Growth Model in basically the same way as in 1951.

8 Conclusion

Rasch worked with his Growth Model for a long period of time, though not with equal intensity. We know for certain that Rasch formulated the Growth Model in 1940, and that this formulation probably was more or less connected to the work he did on growth while studying with Huxley in 1936. We also know for a fact that Rasch from 1940 until the early 1950s occasionally was involved in empirical modelling of growth; most likely as a consequence of that scientists from the medical world consulted him. It is also well documented that Rasch in the early 1950's worked on constructing a solution to the Differential Equation of Growth, but that from the mid 1950s and until the mid 1960's he paid little attention to The Growth Model due to an increasing interest in psychological measurement. When Rasch about 1970 began to use his Growth Model again it was with a view to analyse trends in economic data and to analyse the growth of pigs.

As the years went by Rasch's focus on what theory was most interesting changed. Judging by the Calcutta notes, in 1951, Rasch's main interest was to demonstrate the applicability of the Growth Model to various species. Soon thereafter Rasch's focus changed to that of an interest in stochastic processes. In the late 1950s and in the early 1960s Rasch had little use of his Growth Model, as he was deeply involved in the development of something totally different, namely the concept of specific objectivity. When Rasch took up on the Growth Model again with view to analyse growth in economic data and growth of pigs, he realized that the Growth Model actually fitted inside the concept of specific objectivity, and of course this interested him. For practical purposes, such as analysing the pigs from Lœsø, Rasch used the Growth Model without emphasizing either specific objectivity or the theory for stochastic processes.

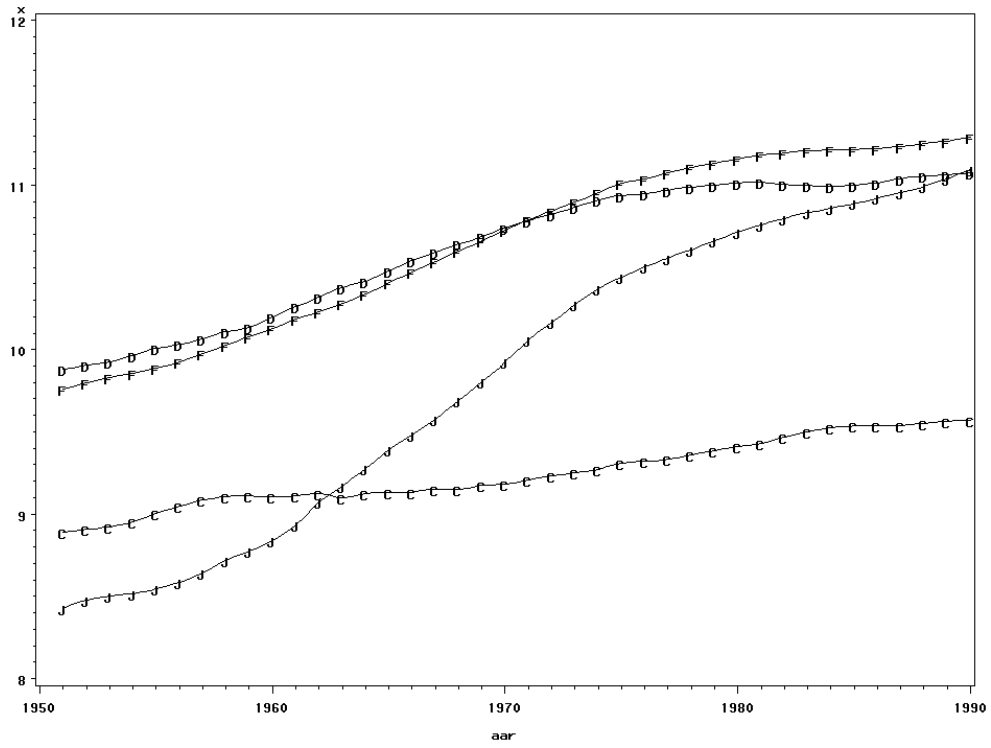


FIGURE 3. *$log(y_\nu(t))$ plotted against time for the four countries, Denmark, France, Japan and Colombia.*

9 Appendix A. Example

In this appendix a small example of using Rasch's Growth Model will be given. The example is about growth in capital per worker for various OECD countries. It must be emphasized that the purpose of this appendix merely is to give an empirical demonstration of applying the Growth Model, and that the chosen data in that sense are arbitrary.

Let $y_\nu(t)$ equal capital per worker,[16] where $t = 1951, 1952, ..., 1990$ and $\nu \in \Omega =$ {*Australia, Austria, Belgium, Brazil, Canada, Colombia, Denmark, Egypt, Finland, France, Germany, Greece, Iceland, India, Ireland, Italy, Japan, Luxembourg, Netherlands, New Zealand, Norway, Pakistan, Paraguay, Portugal, Spain, SriLanka, Sweden, Switzerland, Thailand, Turkey, United Kingdom, United States* }. These countries were chosen because the observations were available throughout the entire period, and because they, with minor exceptions, had positive growth in capital per

[16]Source: Easterly, W. and Ross Levine, "It is not factor accumulation: stylized facts and growth models" , Mimeo, World Bank and U. of Minnesota, September 1999

worker from 1951 to 1990.

First, plots of $(x, y) = (t, \log(y_\nu(t)))_{t=1951,...1990}$ were made for all countries in Ω. A sample of these plots is displayed in Figure 3. Informative as such plots are, it seems difficult to say something general about them, let alone to compare the individual countries based on observations from the entire period. The aim of applying Rasch's Growth Model is to remedy these problems.

As a starting point it was investigated whether the Growth Model applied to all countries in Ω. The metameter, $\tau(t)$, was estimated by the sum of all countries in Ω. (c.f. Page 68):

$$\tau(t) = \sum_{\nu \in \Omega} \log(y_\nu(t), \quad t = 1951, ..., 1990$$

Then, plots of $(x, y) = (\tau(t), log(y_\nu(t)))$ were made. Though some of the plots looked like strait lines, others most certainly did not. This demonstrates that the Growth Model does not apply to all countries at the same time. Neither would this be expected, the point being that the Growth Model states that the growth rate at time t is

$$\frac{y'_\nu(t)}{y_\nu(t)} = log(y_\nu(t))' = \beta_\nu \tau'(t), \tag{2.15}$$

that is, a multiple of a country specific parameter, β_ν, and of a time specific parameter, $\tau'(t)$. But, this implies that that the growth rate, as a function of time, is generated by the same underlying function, namely $\tau'(t)$. In conclusion the Growth Model states that countries progress in basically the same way over time. Considering the diversity of countries in Ω with respect to cultural as well as geographical differences it seems unlikely if this was the case.

On the other hand, it does not seem unlikely that the economics of some countries were influenced by basically the same events; in a specified interval of time that is. Searching for subgroups of countries which progress similarly four groups of countries were disclosed, namely:

- $\Omega_1 = \{$*Australia, Austria, Belgium, Denmark, France, Germany, Greece, Italy, Japan, Netherlands, Portugal, Spain, Switzerland, United Kingdom*$\}$
- $\Omega_2 = \{$*Finland, Iceland, India, Norway, Sweden, Turkey* $\}$
- $\Omega_3 = \{$*Brazil, Columbia, Paraguay*$\}$

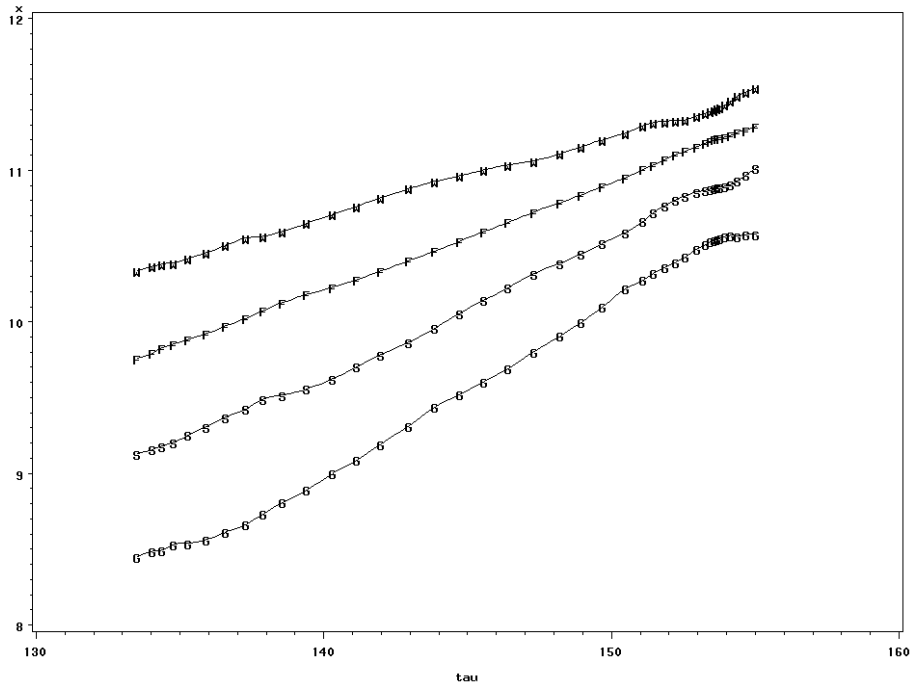


FIGURE 4. *$log(y_\nu(t))$ plotted against $\tau_1(t)$ for respectively F=France, G=Greece, S=Spain and W=Switzerland*

- $\Omega_4 = \{$*New Zealand, United States*$\}$

For each subgroup of countries a metameter was calculated as $\tau_i(t) = \sum_{\nu \in \Omega_i} \log(y_\nu(t))$ $i = 1, 2, 3, 4$. Plots of $(x, y) = (log(y_\nu(t)), \tau_i(t))$ were thereafter made for countries in Ω_i. Figure 4 and Figure 5 are examples of such plots. As is seen, with minor exceptions, the points corresponding to the individual countries scatters around straight lines. Since this is also the case for the remaining countries in the four subgroups, $\Omega_1, ..., \Omega_4$ it has been demonstrated that the Growth Model gives a fair description of the data. The derivations from the model, including specification of the interval of time in which each country is supposed to adhere to the Growth Model, will be commented on in the next Section.

Checking out that no other countries may be included in the four subgroups, $\Omega_1, ..., \Omega_4$, for each $i = 1, 2, 3, 4$, plots of $(x, y) = (\tau_i(t), log(y_\nu(t)))$ were made for the countries $\nu \in \Omega \setminus \Omega_i$. All such plots shows clear derivations from straight lines, which imply that neither of these countries economies progress as a function of time in the same way as is the case for the countries in Ω_i.

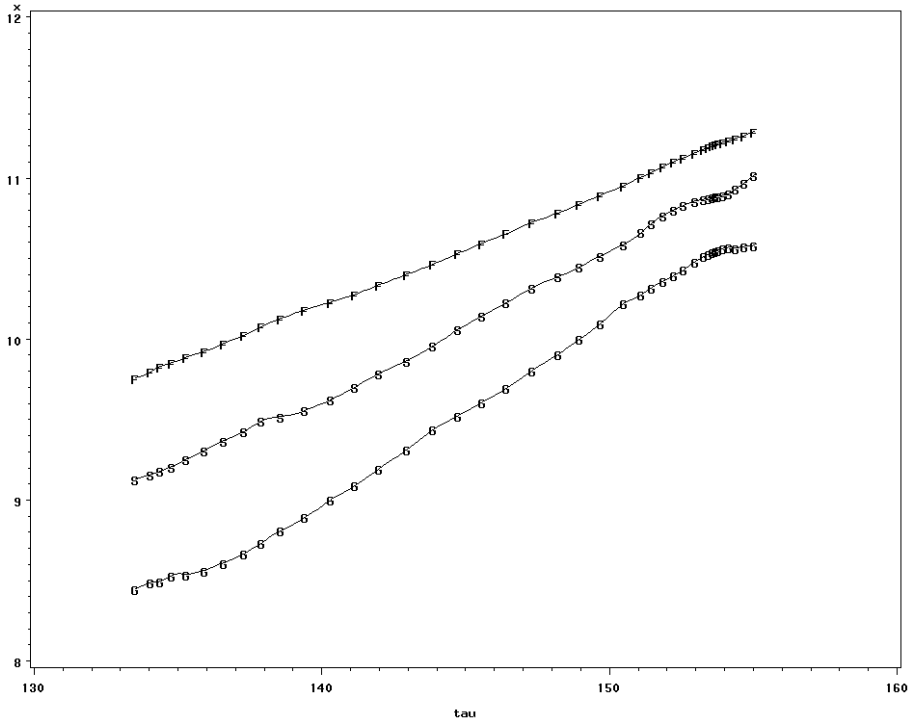


FIGURE 5. *$log(y_\nu(t))$ plotted against $\tau_3(t)$ for respectively B=Brazil, C=Colombia and P=Paraguay*

Now, as a final check that no other subgroups fitting the Growth Model can be found, the observations from the various countries has been plotted pair wise. If two countries, say A and B, fits the Growth Model, a plot of $(x, y) = (log(y_A(t)), log(y_B(t)))$ for $T = 1951, ..., 1990$ will display a straight line. None of these plots looked like straight lines, and in effect it may be concluded that no other subgroup of countries fitting a Growth Model can be found.

Using the Growth Model to gain information

The purpose of using Rasch's Growth Model was to ease the comparison of the various countries. This section will discuss methods to do so. The discussion of these methods will raise questions about the various countries' growth patterns. Since the author is not an economist by training no attempt will be made to answer these questions. The exercise is merely to demonstrate how the Growth Model may be used to extract information out of the data.

Analysing the subgroups

Suppose the Growth Model describes a group of countries. This implies that their progress over time is generated by the same function. But then it is interesting to

have a closer look at which countries constitutes the various subgroups. For instance, what have the countries in Ω_3, that is Brazil, Columbia and Paraguay in common? And does it seem reasonable that their economic progress has followed the same pattern as a function of time? And why is Denmark not in the same group of countries as Norway and Sweden?

In effect, the grouping of countries can be used to gain knowledge about economic growth. It must, however, be stressed that explaining the composition of the various groups is very important. The Growth Model is merely a tool that states that there is empirical evidence that these groups of countries progress similarly. The interesting question is whether this can be justified or not.

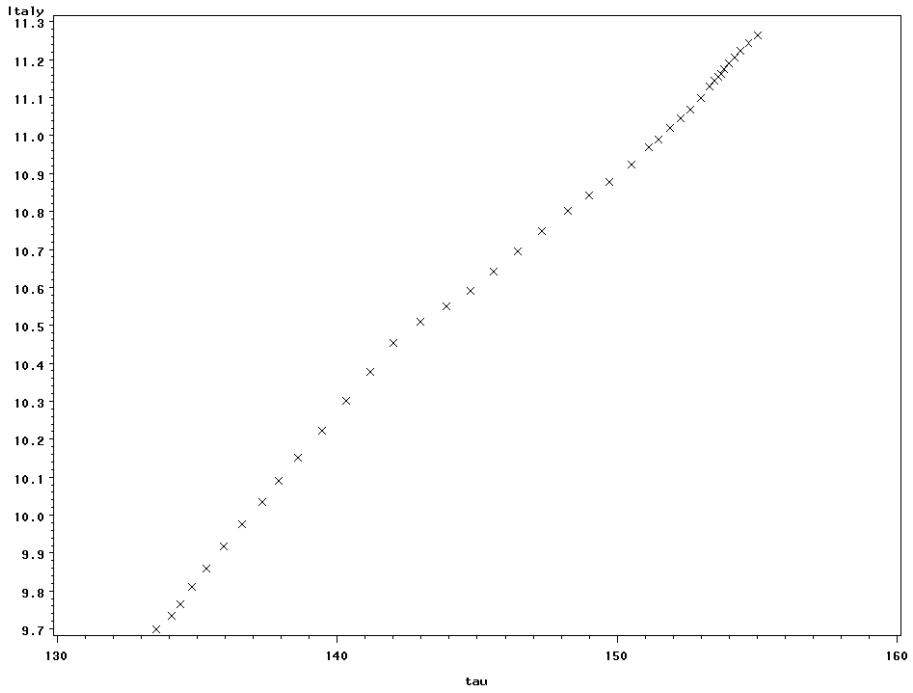


FIGURE 6. *$log(y_\nu(t))$ plotted against $\tau_1(t)$ for Germany.*

Limiting the time intervals

The countries in Ω have been dived into five groups. Four of these groups consist of countries that follow the same growth pattern as a function of time. However, as is seen from the various plots, few countries fit a Growth Model throughout the entire period, 1951 to 1990. For instance, though Japan seems to follow the same growth pattern as do the rest of the countries in Ω_1 this is only the case when $\tau_1(t) > 136$, which corresponds to $t \geq 1957$. One may try to explain why Japan proceeded in the

TABLE 2.2.

Group	Country	Time Interval	Comments
Ω_1	Australia	$[1951, 1990]$	
	Austria	$[1951, 1990]$	
	Belgium	$[1955, 1982]$	
	Denmark	$[1951, 1981]$	
	France	$[1951, 1990]$	
	Germany	$[1951, 1988]$	Different β_ν's for respectively $[1951, 1961]$ and $[1962, 1988]$
	Greece	$[1957, 1986]$	
	Italy	$[1951, 1990]$	Different β_ν's for respectively $[1951, 1963]$ and $[1964, 1990]$
	Japan	$[1957, 1982]$	
	Netherlands	$[1957, 1981]$	
	Portugal	$[1951, 1975]$	
	Spain	$[1951, 1981]$	
	Switzerland	$[1951, 1975]$	
	United Kingdom	$[1951, 1980]$	
Ω_2	Finland	$[1951, 1990]$	
	Iceland	$[1951, 1990]$	
	India	$[1955, 1990]$	Different β_ν's for respectively $[1955, 1977]$ and $[1978, 1990]$
	Norway	$[1957, 1988]$	
	Sweden	$[1951, 1976]$	
	Turkey	$[1957, 1976]$	
Ω_3	Brazil	$[1951, 1978]$	
	Columbia	$[1957, 1982]$	
	Paraguay	$[1957, 1990]$	
Ω_4	New Zealand	$[1951, 1990]$	
	United States	$[1951, 1990]$	

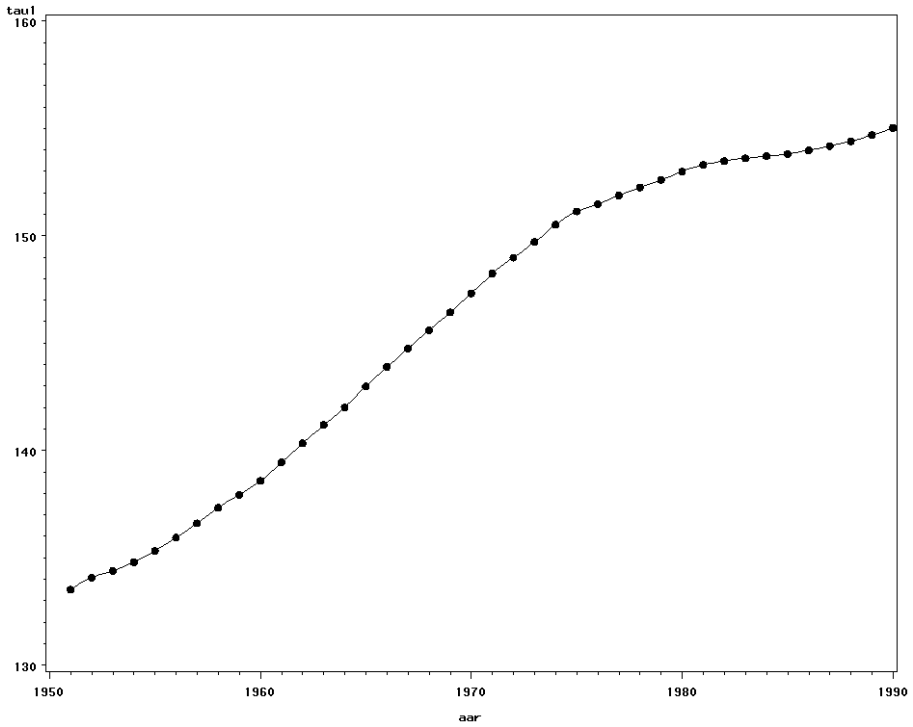


FIGURE 7. $\tau_1(t)$ *plotted against time.*

same way as the remaining countries in Ω_1 in the time interval $[1957, 1982]$, but not in the early 1950s. As another example, a lot of the countries in Ω_1 deviate from the Growth Model for values of τ_1 corresponding to the late 1970s and the 1980s. The only countries that do not exhibit such pattern are *Australia, Austria, France* and *Italy.* This suggests that some events that occurred in the late 1970s and 1980s affected the countries in Ω_1 differently.

In Table 2.2 limits are placed on the interval of time in which each single country can be said to fit a Growth Model. Analysing these limits is a method for extracting information out of the data. Another reason for posing the limits is that any estimation of the country specific parameters, that is the β_ν's, must be based solely on data which do fit the Growth Model. Furthermore, some of the countries exhibit a change of country specific parameter, and these shift must be identified to obtain proper estimates. Figure 6 shows this shift for Germany. Both for the case of Italy and for Germany a shift occurs in the early 1960s. It seems to be an intriguing question why the country specific component of the growth rate for Germany and Italy is larger in the 1950s and early 1960s than is the case for the remaining period.

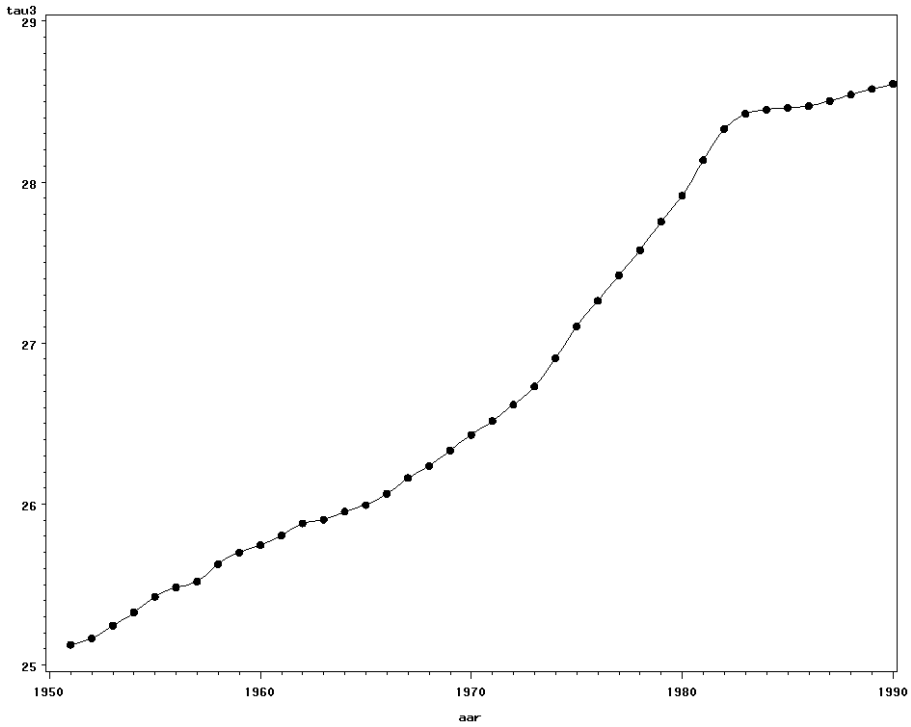


FIGURE 8. $\tau_3(t)$ *plotted against time.*

The metameters

In Figure 7 and Figure 8 the metameters, $\tau_1(t)$ and $\tau_3(t)$ are displayed as a function of time. Since the time specific component of the growth rate is $\tau_i'(t)$ the steepness of these graphs pictures the time specific component of the growth rate. The metameters can therefore be used to gain knowledge about the economic progress of the different groups of countries. Of course, such analysis presumes that the grouping of countries makes sense from an economic point of view.

Comparing the β_ν's

Within each group of countries, $\Omega_1, ..., \Omega_4$, the country specific parameters may be used to compare the countries. Details about the estimation of these parameters are given in the next Section.

The β_ν's are graphically illustrated by the slope of the straight lines which appear when a plot is made of $(x, y) = (\tau_i(t), log(y_\nu(t)))$ for country ν in Ω_i. However, since the x-axis is transformed by the metameter, $\tau_i(t)$, little information is gained by just looking at the size of the country specific parameters. Instead, it must be compared to other country specific parameters. In this respect it seems useful to transform the country specific parameters into an index as follows: At each point in time we have

that growth rate of country A relative to country B is

$$\frac{y'_A(t)/y_A(t)}{y'_B(t)/y_B(t)} = \frac{\beta_A \tau'(t)}{\beta_B \tau'(t)} = \frac{\beta_A}{\beta_B}. \tag{2.16}$$

But then we may choose any country, say country A_i, in Ω_i, $i = 1, 2, 3, 4$ as our point of reference by defining this country as 100, that is, $\vartheta_{A_i} = 100$. The corresponding number for the remaining countries in Ω_i is then calculated from Equation (5.8) as $\vartheta_B = 100\beta_B/\beta_{A_i}$. The estimates of the β_ν's and the corresponding index are given in Table 2.3.

Estimating the country specific parameters

In this Section the β_ν's will be estimated, and the assumptions that these estimates are based on will be validated.

The estimate of the β_ν's that Rasch suggested in Equation 2.8 has been calculated. They are listed in Table 2.3 under 'R'. These estimates have further been transformed into an index. These estimates can be found in Table 2.3, under 'R-I'.

Rasch's approach for constructing a stochastic process is without doubt proper. But, it is still interesting to try solving the Difference Equation as we would do today: The Differential Equation of Growth may be written as

$$dY_t = \beta\tau'(t)Y_t dt + \sigma\sqrt{\tau'(t)}Y_t dZ_t \tag{2.17}$$

or

$$\frac{dY_t}{Y_t} = \beta\tau'(t)dt + \sigma\sqrt{\tau'(t)}dZ_t.$$

where Z_t is a Brownian motion. Though it might seem reasonable that the noise of Equation (2.17), namely $\sigma\sqrt{\tau'(t)}Y_t dZ_t$, includes the metameter, it might seem forced to choose the functional form, $\sqrt{\tau'(t)}$. However, this gives a nice solution, and, as we shall see, the model turns out to describe the data quite well.

Using Ito's lemma on $g(t, Y_t) = \log(Y_t)$ we have (see e.g. Øksendal (1998, p. 44))

$$\begin{aligned} d\log(Y_t) &= \frac{\partial g(t,Y_t)}{\partial t}dt + \frac{\partial g(t,Y_t)}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 g(t,Y_t)}{\partial y^2}(dY_t)^2 \\ &= \frac{dY_t}{Y_t} - \frac{(dY_t)^2}{2Y_t^2} = \beta\tau'(t)dt + \sigma\sqrt{\tau'(t)}dZ_t - \frac{(dY_t)^2}{2Y_t^2} \end{aligned}$$

Now, since $dt^2 = dtdZ_t = 0$ and $dZ_t^2 = dt$ (Øksendal, 1998, p. 44), we have that

$$\begin{aligned} (dY_t)^2/Y_t^2 &= \left(\beta^2\tau'(t)^2Y_t^2dt^2 + \tau'(t)\sigma^2Y_t^2(dZ_t)^2 + 2\beta\tau'(t)\sigma Y_t^2 dtdZ_t\right)/Y_t^2 \\ &= \tau'(t)\sigma^2 dt \end{aligned}$$

TABLE 2.3. **Estimates**

Group	Country	Time Interval	R	R-I	SDE-I	SDE
Ω_1	Australia	$[1951, 1990]$	0.0351	100	100	0.0352
	Austria	$[1951, 1990]$	0.0866	246.8	246.1	0.0867
	Belgium	$[1955, 1982]$	0.0866	153.2	152.8	0.0539
	Denmark	$[1951, 1981]$	0.0574	163.6	163.2	0.0575
	France	$[1951, 1990]$	0.0712	202.9	202.3	0.0712
	Germany	$[1951, 1961]$	0.0882	251.3	250.6	0.0882
	Germany	$[1962, 1988]$	0.0615	175.1	174.6	0.0615
	Greece	$[1957, 1986]$	0.1123	320.1	319.3	0.1124
	Italy	$[1951, 1963]$	0.0885	252.1	251.2	0.0885
	Italy	$[1964, 1990]$	0.0622	177.4	176.9	0.0623
	Japan	$[1957, 1982]$	0.1276	363.6	362.8	0.1278
	Netherlands	$[1957, 1981]$	0.0520	148.3	147.9	0.0521
	Portugal	$[1951, 1975]$	0.0816	232.5	232.2	0.0818
	Spain	$[1951, 1981]$	0.0879	250.6	250.2	0.0881
	Switzerland	$[1951, 1975]$	0.0546	155.6	155.3	0.0545
	England	$[1951, 1980]$	0.0528	150.6	150.1	0.0529
Ω_2	Finland	$[1951, 1990]$	0.1825	100	100	0.1828
	Iceland	$[1951, 1990]$	0.1525	83.6	83.7	0.1530
	India	$[1955, 1977]$	0.1485	81.3	81.3	0.1486
	India	$[1978, 1990]$	0.2420	132.6	132.6	0.2424
	Norway	$[1957, 1988]$	0.1333	73.0	73.0	0.1335
	Sweden	$[1951, 1976]$	0.1430	78.4	78.3	0.1432
	Turkey	$[1957, 1976]$	0.2565	140.5	140.5	0.2568
Ω_3	Brazil	$[1951, 1978]$	0.3932	100	100	0.3943
	Columbia	$[1957, 1982]$	0.1358	34.5	34.8	0.1371
	Paraguay	$[1957, 1990]$	0.5194	132.4	132.0	0.5204
Ω_4	New Zealand	$[1951, 1990]$	0.4754	100	100	0.4856
	United States	$[1951, 1990]$	0.5246	110.4	106.5	0.5174

TABLE 2.4. **QQ plots, R^2 and Durbin Watson test.** Regarding the QQ plots, +, stands for no sign of derivation from the normal distribution, ? means that it is undecided, and − means that the observations cannot be described by a normal distribution. For the case of the Durbin Watson test, + stands for no auto correlation, ? stands for undecided, and − imply that the students residuals are auto correlated.

Group	Country	Time Interval	QQ	R^2	DW
Ω_1	Australia	[1951, 1990]	+	0.7309	+
	Austria	[1951, 1990]	+	0.9573	?
	Belgium	[1955, 1982]	+	0.9616	?
	Denmark	[1951, 1981]	+	0.9297	?
	France	[1951, 1990]	+	0.9785	-
	Germany	[1951, 1961]	?	0.9677	+
	Germany	[1962, 1988]	+	0.9626	+
	Greece	[1957, 1986]	?	0.9730	+
	Italy	[1951, 1963]	+	0.9874	+
	Italy	[1964, 1990]	+	0.9557	-
	Japan	[1957, 1982]	+	0.9726	-
	Netherlands	[1957, 1981]	?	0.9568	-
	Portugal	[1951, 1975]	?	0.9319	+
	Spain	[1951, 1981]	+	0.9319	-
	Switzerland	[1951, 1975]	-	0.9331	?
	England	[1951, 1980]	-	0.9785	-
Ω_2	Finland	[1951, 1990]	+	0.9165	-
	Iceland	[1951, 1990]	+	0.8241	+
	India	[1955, 1977]	-	0.9418	+
	India	[1978, 1990]	-	0.9023	+
	Norway	[1957, 1988]	-	0.9150	-
	Sweden	[1951, 1976]	+	0.9451	-
	Turkey	[1957, 1976]	+	0.9612	+
Ω_3	Brazil	[1951, 1978]	?	0.8696	+
	Columbia	[1957, 1982]	+	0.4492	?
	Paraguay	[1957, 1990]	+	0.9291	-
Ω_4	New Zealand	[1951, 1990]	+	0.7052	-
	United States	[1951, 1990]	+	0.7306	-

which imply that

$$d\log(Y_t) = (\beta - \frac{1}{2}\sigma^2)\tau'(t)dt + \sigma\sqrt{\tau'(t)}dZ_t$$

But then, an estimate of β can be obtained by estimating μ in

$$dW_t = d\log(Y_t) = \mu\tau'(t)dt + \sigma\sqrt{\tau'(t)}dZ_t \tag{2.18}$$

and then calculate β as $\beta = \mu + \frac{1}{2}\sigma^2$. This assumes that σ^2 has been estimated.

By the Ito interpretation equation (2.18) is equivalent to

$$\begin{aligned} W_t &= W(0) + \mu \int_0^t \tau'(s)ds + \sigma \int_0^t \sqrt{\tau'(t)}dZ_s \\ &= W(0) + \mu(\tau(t) - \tau(0)) + \sigma \int_0^t \sqrt{\tau'(t)}dZ_s, \end{aligned}$$

where the Ito integral, $\int_0^t \sqrt{\tau'(t)}dZ_s$, is normally distributed with mean zero and variance $\int_0^t \tau'(s)ds = \tau(t) - \tau(0)$.

We shall find the distribution of the the successive differences

$$\begin{aligned} & W_{t_i} - W_{t_{i-1}} \\ = \; & \mu(\tau(t_i) - \tau(t_{i-1})) + \sigma(\int_0^{t_i} \sqrt{\tau'(t)}dZ(s) - \int_0^{t_{i-1}} \sqrt{\tau'(t)}dZ_s) \\ = \; & \mu(\tau(t_i) - \tau(t_{i-1})) + \sigma \int_{t_{i-1}}^{t_i} \sqrt{\tau'(t)}dZ_s. \end{aligned}$$

Since the $\int_{t_{i-1}}^{t_i} \sqrt{\tau'(t)}dZ_s$'s, $i = 1, ..., n$, are Ito integrals, they are independent, normally distributed with mean zero and variance $\tau(t_i) - \tau(t_{i-1})$. Hence, the $W_{t_i} - W_{t_{i-1}}$'s are normally distributed with mean $\mu(\tau(t_i) - \tau(t_{i-1}))$ and variance $\sigma^2(\tau(t_i) - \tau(t_{i-1}))$.

The result obtained by finding a solution to the stochastic differential equation (2.17) is therefore quite similar to Rasch's solution. Rasch constructed a piecewise deterministic process, but by approaching the limiting case $\lambda \to \infty$, he got that the variables

$$\log(y(t_j)) - \log(y(t_{j-1})) \tag{2.19}$$

are independent, normal distributed with mean $\beta(\tau(t_j) - \tau(t_{j-1}))$ and variance $\sigma^2(\tau(t_j) - \tau(t_{j-1}))$, $j = 1, ..., n$. On the other hand, departing from the stochastic differential equation (2.17) we get that the variables, $\log(Y_{t_j}) - \log(Y_{t_{j-1}})$ are normally distributed with mean $(\beta - \frac{1}{2}\sigma^2)(\tau(t_j) - \tau(t_{j-1}))$ and variance $\sigma^2(\tau(t_j) - \tau(t_{j-1}))$, $j = 1, ..., n$, where the correction $\frac{1}{2}\sigma^2$ is due to the fact that we started by considering the process Y_t and then transformed the problem such that we consider the process $\log Y_t$ instead. Rasch does not make such a shift.

Since the distributions are quite similar, so are the estimates. Actually, we only need to add $\frac{1}{2}\sigma^2$ to the estimates we got by using Rasch's approach, and, as it is, the variance is relatively small. The estimates of the β_ν's obtained by departing from (2.17) are given in Table 2.3 under 'SDE' and the corresponding index can be found under 'SDE-I'.

As to the control of the assumptions made, in both cases we may check whether the $\log(y(t_j)) - \log(y(t_{j-1}))$ are normally distributed with mean $\theta(\tau(t_j) - \tau(t_{j-1}))$ and variance $\sigma^2(\tau(t_j) - \tau(t_{j-1}))$ by noticing that this resemble a model for a weighted regression with no intercept. The standard ways of checking such a model have been applied. For each regression the students residuals were plotted against the explanatory variables and against the predicted variables. The Durbin Watson test for auto correlation was performed, and by making QQ plots[17] it was validated whether the $\log(y(t_j)) - \log(y(t_{j-1}))$ are in fact normally distributed. Table summarizes some of the result. Though some countries fit better to the model than others, in general it seems that the proposed model is adequate to describe the data.

10 Appendix B. Interviews

Appendix B contains background information about the people who so kindly told the present author about their memories of Rasch's Growth Model. Special emphasis is on their connection to Rasch.

Allerup, Peter. Interviews 26.02.1998 and 26.02.1998. Peter Allerup is a statistician by training. He became associated with Rasch in the 1960s, and he worked with him at both the Statistical Institute and the Danish Institute of Educational Research, where Rasch was affiliated as a consultant. Allerup has continued to work with the Rasch Models at the Danish University of Educational Research, where he now holds a position as professor of statistics.

Andrich, David. Interview. 04.02.2002 and 06.02.2002, Perth, Western Australia. Andrich met Rasch in 1972 in Chicago. At this point in time, Andrich was a graduate student, and Rasch was newly retired. Rasch was visiting Ben Wright in Chicago, and Wright, who was the Chair of Andrich's PhD committee, persuaded Andrich that, he on his way back to Australia in 1973, he should visit Rasch in Denmark. Andrich stayed with Rasch for a week or so, and in 1974 Andrich arranged for Rasch to be a visiting professor for seven months in the Departments of Mathematics and Education at the University of Western Australia, where Andrich held a position. Andrich visited Rasch again in 1975, and in 1977 Andrich spent five months at the Danish Institute of Educational Research, where Rasch was still affiliated as a consultant. In 1979, Andrich visited Rasch with the purpose of interviewing him. The interview Andrich conducted

[17] In Danish: normalfordelingsfraktildiagrammer

on this occasion is the one referred to throughout this article as the Andrich-Interview. Andrich is now professor of Education at Murdoch University in Western Australia.

Niels-Erik Jensen Interview 27.06.2001. Niels-Erik Jensen became affiliated with the Statistical Institute in the mid 1960s as an tutor. Later Rasch employed him to give lectures in a course on difference and differential equations. In 1968 Niels-Erik Jensen was appointed to one of the permanent positions available at the Statistical Institute. He soon became involved in the administrative duties, and became associate professor with administrative responsibilities[18] in 1971. Niels-Erik Jensen still (in 2002) holds a position as associate professor, though the Statistical Institute today comes under the Institute of Economics.

Stene, Jon. Interview 08.10.2002 and 29.10.2002. In 1963, Stene was employed at the Danish Institute of Educational Research. As a statistical consultant, Rasch worked at the Danish Institute of Educational Research every Friday, and it was here that Stene got to know Rasch. Stene became affiliated with the Statistical Institute in 1964 as a tutor, and in 1966, he got a position at the Statistical Institute. He soon became deeply involved in teaching and the writing of Rasch's second textbook. Furthermore, most of the unpublished papers 'written' by Rasch on the Models for Measurement have been thoroughly prepared by Stene. Today Stene is retired, but he held a position at the Statistical Institute until 2000 (In 1996 the Institute of Economics incorporated the Statistical Institute, implying that technically speaking, Stene held a position at the Institute of Economics from 1996 to 2000)

11 Appendix C. Quotes

Appendix B contains all quotes in which the original was in Danish and which have been translated to English in the paper.

The original text to the quote on Page 65: Det er velkendt at saa simple Organismer som f.Ex. Colibakterier under optimale Vækstforhold formerer sig efter en saare simpel multiplikativ Lov: i Løbet af en vis Tidsenhed bliver een Bakt. til 2, i Løbet af den næste bliver 2 til 4 o.s.v., saaledes bliver det ved saalænge Miliøet ikke bliver forgiftet og saa længe der er Næring nok til alle Bakterierne. Enhver Vækst sker jo

[18] In Danish: afdelingsleder.

ved Celledelinger, for højerestaaende Organismer selvfølgelig af en langt mere kompliceret Natur end for Colibakterier, men Princippet er i og for sig det samme, men netop naar det drejer sig om højerestaaende Organismer, som f.Ex. Kalve, spiller "Alderen", med alt det den indebærer af Ændringer i det indre og ydre Miljø, en afgørende Rolle. Men saa er det fristende, at opfatte det grundlæggende Vækstprincip som i og for sig det samme, blot at Kalendertidsenhederne paa nogle Stadier er lang mere vækstværdifulde end paa andre - altsaa ligefrem at opretholde den simple elementære Vækstlov, den relativt konstante Tilvækst pr. Tidsenhed, naar man bare fandt de fysiologisk adækvate Tidsenheder at udtrykke Tiden i. Resultatet af min Beskæftigelse med Deres Talmateriale har været mig en stor Opmuntring i den Retning. De kan fortolkes paa den Maade, at - indenfor det begrænsede Tidsrum $0.5 - 20.5$ Mrd. - den nævnte simple Vækstlov virkelig gælder for enhver af Deres Kalve m.H.t. alle de 6 Maal De har taget og det uanset, hvordan de har været fodrede, naar man som fysiologisk Maal for Alderen anvender Logaritmen til den kronologiske Alder plus 2 Mrd.

The original text to the quote on Page 77: At De S. 63 tegner nogle Kurver over Forløbet af disse Tilvækster hjælper ikke ret meget paa ens principielle Usikkerhed overfor det Faktum, at de Tal jo maa hænge sammen som Ærtehalm paa en eller anden mystisk Maade - at man altsaa ikke kan vurdere Forskellen mellem Holdene paa et Tidspunkt og paa et andet Tidspunkt uden at maatte geraade ud i de frygteligste statistiske Dilemmaer.

The original text to the quote on Page 77: Er man først i Besiddelse af dette Princip, ja saa giver Resten sig jo næsten af sig selv. Hvert enkelt Dyrs f.Ex. Vægtkurve er da paa nær mer eller mindre tilfældige Afvigelser bestemt af 2 Konstanter [...] Med dette Vaaben i Hænde kan man endelig give sig i Lag med bogstavelig talt hvilke som helst af de Undersøgelser, som de har ønkset at foretage, men som De har savnet Midlerne til.

12 References

Andersen, B. (1980). Georg Rasch. *Årbog for Københavns Universitet. 1980-81*, pp. 536−540. Denmark

Andersen, E.B. and Olsen, L.W. (2001). The Life of Georg Rasch as a Mathematician and as a Statistician. In Boomsma, A., van Duijn, M.A.J., Snijders, T. (Ed.)*Essays on Item Response Theory.* Springer Verlag.

Andersen, B. (1995). What Georg Rasch Would Have Thought about this Book. In Fischer, G.H. and Molenaar, (Ed.) I.W. *Rasch Models.* New-York: Springer-Verlag, pp. 383−390.

Davis, M.H.A (1984). Piecewise-Deterministic Markov Processes: A General Class of Non-Diffusion Stochastic Models. *Journal of the Royal Statistical Society*, 46, pp. 353−388.

Huxley, J.s. (1932). *Problems of Relative Growth.* Methuen and Co: London.

Izenman, A.J.,Williams, J.S. (1989) A class of linear spectral models and analysis for the study of longitudinal data. *Biometrics,* 45, Issue 3, pp. 831−849.

Rao, C.S. (1958). Some Statistical Methods for comparison of growth curves. *Biometrics,* 14, pp. 1−17.

Rasch, G. (1930). Om Matrixregning og dens Anvendelser paa Differens- og Differentialligninger. Copenhagen: Levin and Munksgaars forlag.

Rasch, G. (1950). Statistik og forsøg. Orientering i et materiale af markforsøg over kvælstofgødninger. *Tidsskrift for Planteavl,* 53, 369−401.

Rasch, G. (1954). *Contributions to the Theory of Stochastic Differential Equations.* Typewritten. Duplicated. Copenhagen.

Rasch, G. (1960). *Probabilistic Models for Some Intelligence and Attainment Tests.* Copenhagen: The Danish Institute of Educational Research.

Rasch, G. (1972). Objektivitet i Samfundsvidenskaberne et metodeproblem. *Nationaløkonomisk tidsskrift,* 110, 161−196.

Rasch, G. (1977). On specific objectivity. An attempt at formalizing the request for generality and validity of scientific statements. In M. Blegvad (Ed.), *The Danish Yearbook of Philosophy*, (pp. 58−94). Copenhagen: Munksgaard.

Rasch, G. (1980). *Probabilistic Models for Some Intelligence and Attainment Tests.* Expanded edition. Chicago: The University of Chicago Press.

Øksendal, B. (1998). *Stochastic Differential Equations. An Introcuction with Appli-*

cations. Fifth Edition. Springer Verlag

13 After word by L.W.O

After Andersen and Olsen submitted *The life of Georg Rasch as a mathematician and as a statistician* for publication I have gotten access to the papers and books that Rasch had in his home office (for further references: Rasch's library). I went through this rather extensive amount of material hoping to find out more about Rasch's analysis of the slow readers (c.f. Section 4). More specifically, I was hoping to find the report on this analysis that Rasch, together with M. Ellehammer and Carl Aage Larsen, made for the Danish Ministry of Social Affairs (Rasch, 1952). I did find a note in Rasch's library saying that one of Rasch's colleagues at DPI had borrowed

the report, Rasch (1952), in 1963. But, when this colleague was confronted with the note, he claimed that he knew nothing about it. Furthermore, though asking libraries, likely ministries and 'colleagues' of Rasch.[2] it has not been possible to locate the report, Rasch (1952).

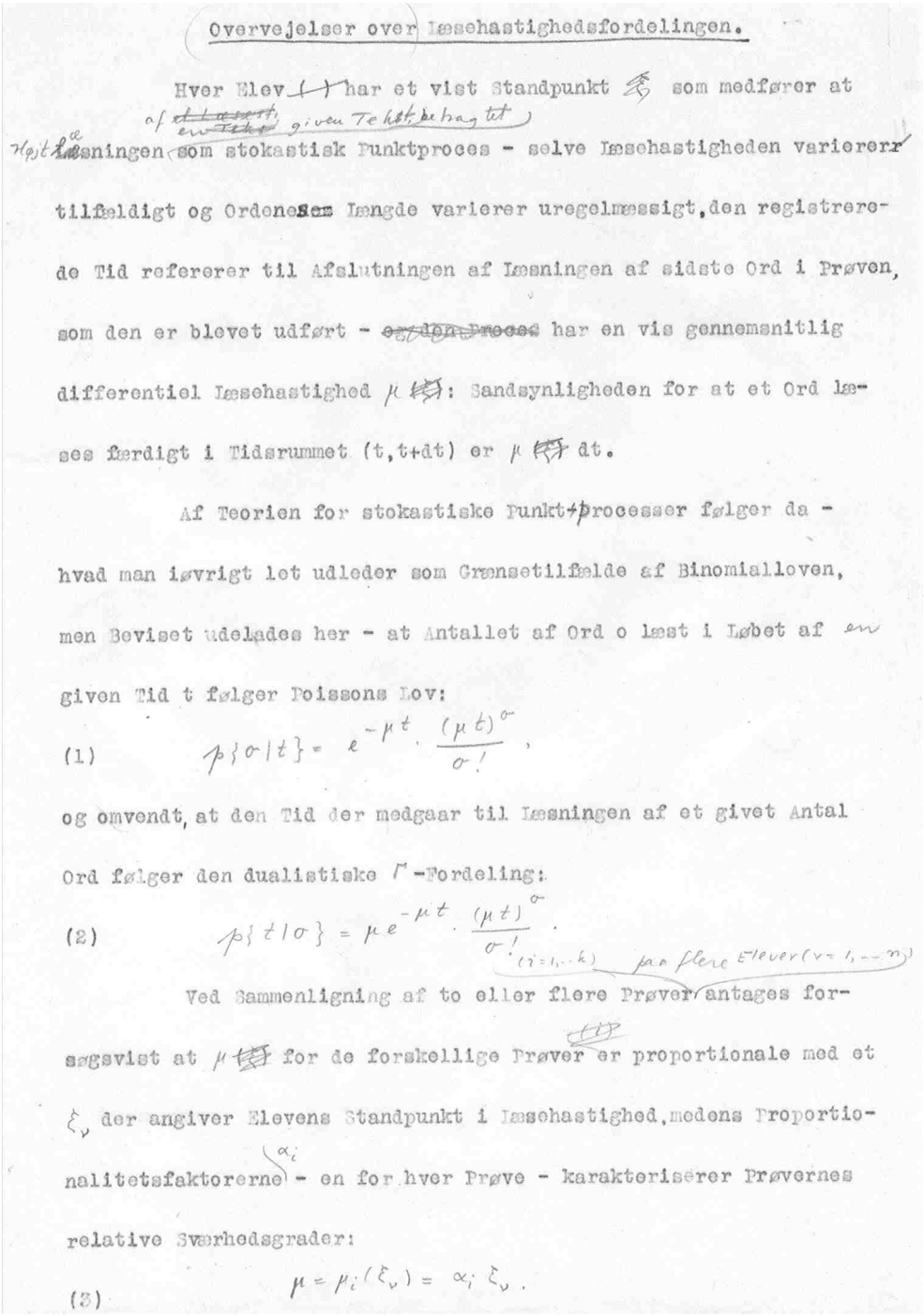
Fortunately, Rasch's library contains a rough draft of Rasch (1952). For further references we shall call this *The Draft*. The Draft consists of a great number of calculations and plots, mixed with hand written notes and a few typewritten pages. There is no telling of the completeness of The Draft, but it is, nonetheless, quite informative. Some of these interesting historical findings are reproduced and explored here.

Figure 6 is a scanning of The Draft that proves that Rasch used both a multiplicative Poisson model and a Gamma model in his analysis of the slow readers (which, by the way, suffered from dyslexia). This is probably the first place where Rasch wrote down these two models that are so essential in his 1960 book (Rasch, 1960).

More importantly, Figures 7 and 8 document that Rasch from the very beginning estimated the item parameters in the conditional distributions that are independent of the person parameters. Furthermore, equation (3.5) of Figure 7 substantiates that Rasch used bridge-building in his analysis of the slow readers, that is, after having estimated the item parameters, he estimated the person parameters by using the estimated item parameters. Yet an intriguing aspect is evident in Figure 8, namely that Rasch already at this point in time found it important that the multiplicative Poisson model fulfilled what Rasch later on termed a *separability theorem* (c.f. Page 115). Finally, Figure 9 is a demonstration that Rasch actually had generalized his ideas to apply to several items as well as several individuals at the same time; essentially as in his 1960 book (Rasch, 1960). In conclusion, with regards to the multiplicative Poisson model it seems that Rasch's ideas on bridge-building and conditional inference was quite advanced in 1952.

Turning to Rasch's development of the Rasch models, as is seen from Figure 10 Equation (1.4), Rasch did write down the mathematical form of the dichotomous Rasch model in the Draft. This must, however, not be construed to mean that this model was used to analyse the slow readers. There is no indication whatsoever that this was the case. On the contrary, Figure 11 shows Rasch's reason for including the

[2]For instance Peter Allerup, Erling Andersen, Børge Prien, Jon Stene.



Overvejelser over Læsehastighedsfordelingen.

Hver Elev (i) har et vist Standpunkt ξ_ν som medfører at af en given Tekst Højt Læsningen som stokastisk Punktproces - selve Læsehastigheden varierer tilfældigt og Ordenes Længde varierer uregelmæssigt, den registrerede Tid refererer til Afslutningen af Læsningen af sidste Ord i Prøven, som den er blevet udført - en sådan Proces har en vis gennemsnitlig differentiel Læsehastighed μ ξ_ν: Sandsynligheden for at et Ord læses færdigt i Tidsrummet (t, t+dt) er μ ξ_ν dt.

Af Teorien for stokastiske Punktprocesser følger da - hvad man iøvrigt let udleder som Grænsetilfælde af Binomialloven, men Beviset udelades her - at Antallet af Ord o læst i Løbet af en given Tid t følger Poissons Lov:

(1) $$p\{o \mid t\} = e^{-\mu t} \cdot \frac{(\mu t)^o}{o!},$$

og omvendt, at den Tid der medgaar til Læsningen af et givet Antal Ord følger den dualistiske Γ-Fordeling:

(2) $$p\{t \mid o\} = \mu e^{-\mu t} \cdot \frac{(\mu t)^o}{o!}.$$

$(i = 1, \ldots k)$ paa flere Elever $(\nu = 1, \ldots n)$

Ved Sammenligning af to eller flere Prøver antages forsøgsvist at μ ξ_ν for de forskellige Prøver er proportionale med et ξ_ν der angiver Elevens Standpunkt i Læsehastighed, medens Proportionalitetsfaktorerne α_i - en for hver Prøve - karakteriserer Prøvernes relative Sværhedsgrader:

(3) $$\mu = \mu_i(\xi_\nu) = \alpha_i \xi_\nu.$$

FIGURE 6. Scanning of the Draft. A multiplicative Poisson model and a Gamma model. In both models, each person and each item is parameterized (the $\xi_\nu' s$ and the α_i's respectively). The text of this Figure has been translated to English in Appendix A.

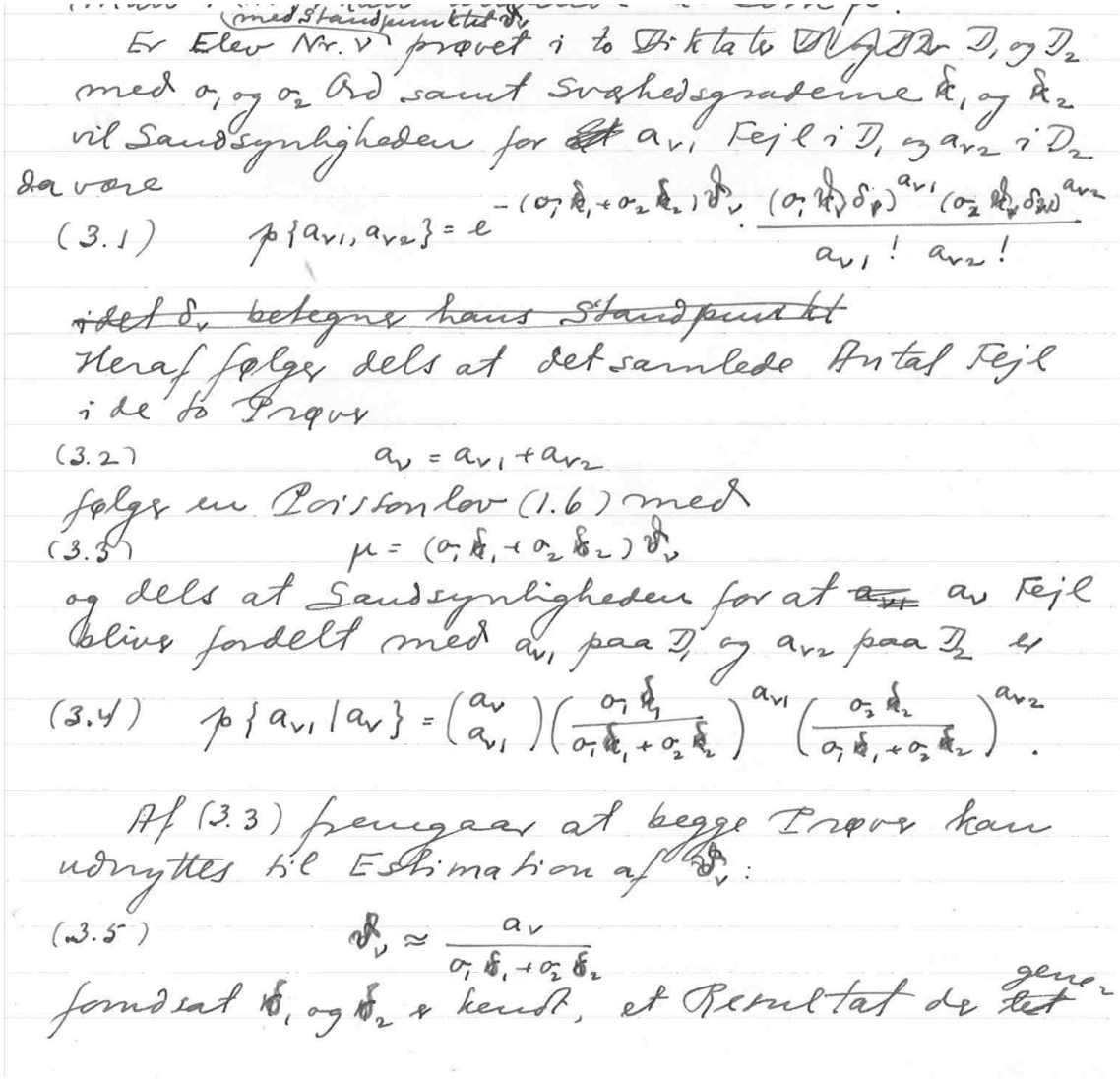


FIGURE 7. Scanning from the Draft. The Figure is continued in Figure 8. Rasch's formula (3.5) substantiates that he used bridge-building. The text has been translated to English in Appendix A.

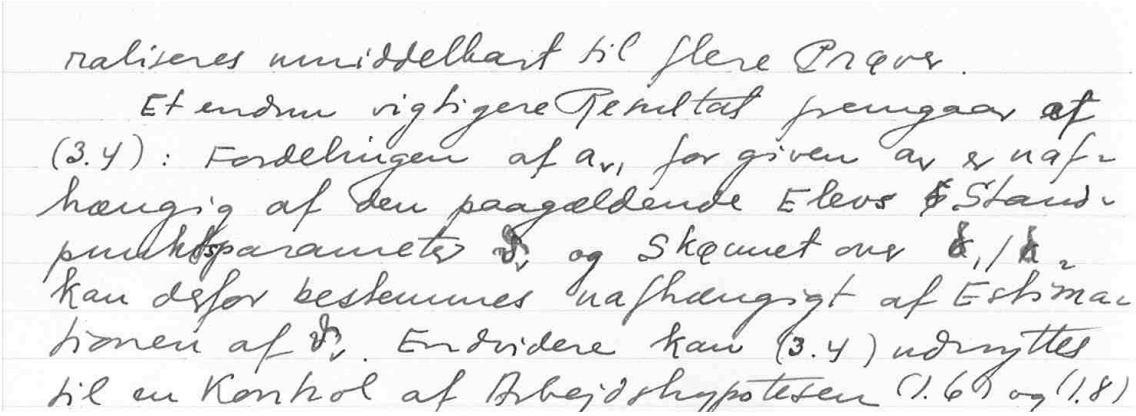


FIGURE 8. This Figure in a continuation of Figure 7. The two Figures, 7 and 8, proves that Rasch suggested to 'eliminate nuisance parameters' in the multiplicative Poisson model by conditioning on the sufficient statistics, and then estimate in the conditional distribution. The text has been translated to English in Appendix A.

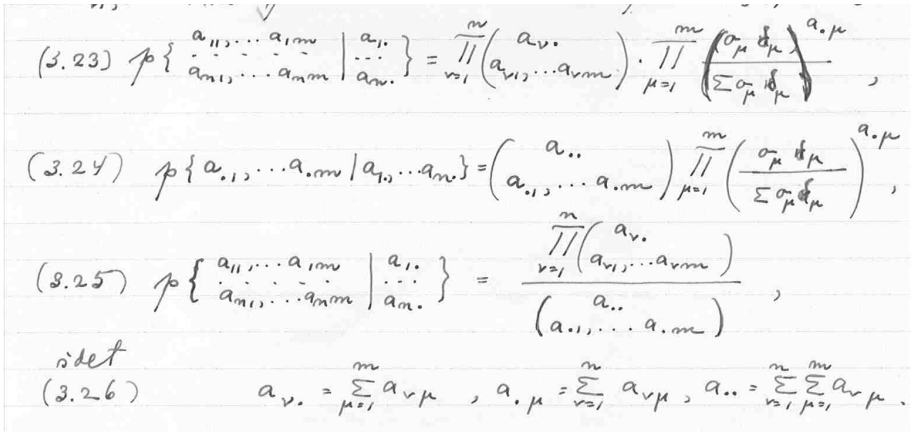
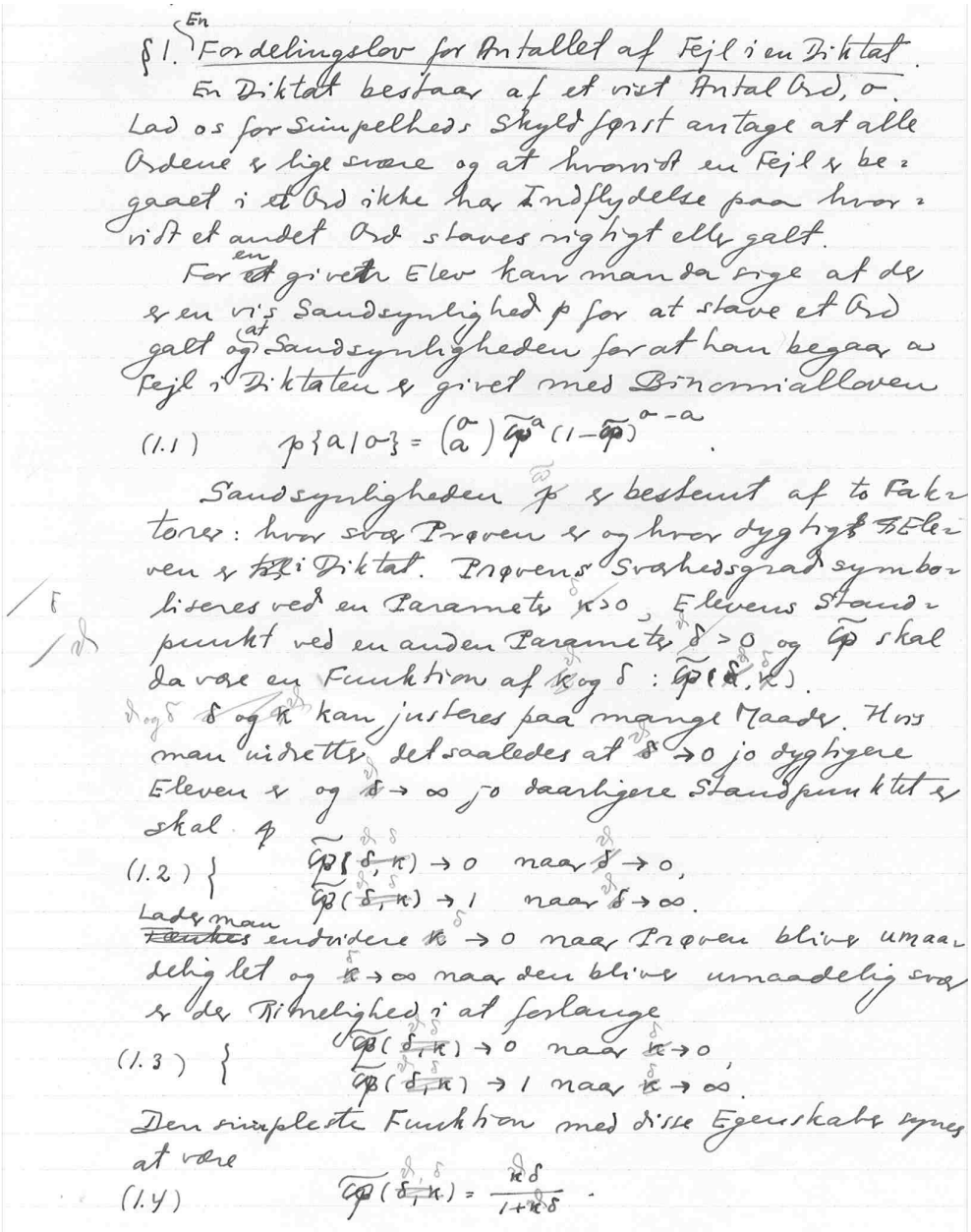


FIGURE 9. Conditional distributions. Draft to (Rasch, 1952). Essentially the same derivations of conditional distributions appears in Rasch (1960, ch. VIII, ch. IX.). In equation (3.25) a conditioning on $(a_{.1}, ..., a_{.n})$ is lacking.



§ 1. En Fordelingslov for Antallet af Fejl i en Diktat.

En Diktat bestaar af et vist Antal Ord, o. Lad os for Simpelheds Skyld først antage at alle Ordene er lige svære og at hvorvidt en Fejl er begaaet i et Ord ikke har Indflydelse paa hvorvidt et andet Ord staves rigtigt eller galt.

For en given Elev kan man da sige at der er en vis Sandsynlighed p for at stave et Ord rigtigt og Sandsynligheden for at han begaar a Fejl i Diktaten er givet ved Binomialloven

$$(1.1) \qquad p\{a \mid o\} = \binom{o}{a} \varphi^a (1-\varphi)^{o-a}.$$

Sandsynligheden p er bestemt af to Faktorer: hvor svær Prøven er og hvor dygtig Eleven er til Diktat. Prøvens Svarhedsgrad symboliseres ved en Parameter $\kappa > 0$, Elevens Standpunkt ved en anden Parameter $\delta > 0$ og φ skal da være en Funktion af κ og δ: $\varphi(\delta, \kappa)$.

δ og κ kan justeres paa mange Maader. Hvis man vil rette det saaledes at $\delta \to 0$ jo dygtigere Eleven er og $\delta \to \infty$ jo daarligere Standpunktet er skal φ

$$(1.2) \quad \begin{cases} \varphi(\delta, \kappa) \to 0 & \text{naar } \delta \to 0, \\ \varphi(\delta, \kappa) \to 1 & \text{naar } \delta \to \infty. \end{cases}$$

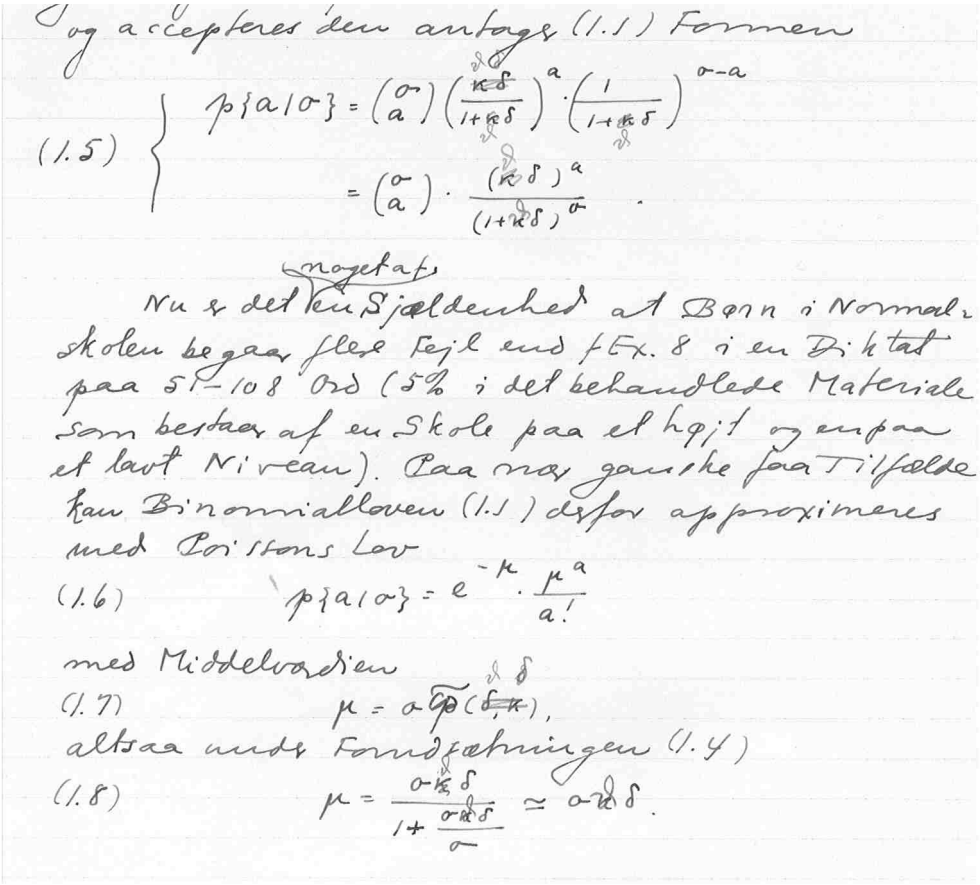
Lader man endvidere $\kappa \to 0$ naar Prøven bliver umaadelig let og $\kappa \to \infty$ naar den bliver umaadelig svær er der Rimelighed i at forlange

$$(1.3) \quad \begin{cases} \varphi(\delta, \kappa) \to 0 & \text{naar } \kappa \to 0, \\ \varphi(\delta, \kappa) \to 1 & \text{naar } \kappa \to \infty. \end{cases}$$

Den simpleste Funktion med disse Egenskaber synes at være

$$(1.4) \qquad \varphi(\delta, \kappa) = \frac{\kappa\delta}{1+\kappa\delta}.$$

FIGURE 10. Scanning from the Draft. Equation (1.4) resemblance the dichotomous Rasch model. This figure is continued in Figure 11. The text has been translated to English in Appendix A.



Og accepteres den antagne (1.1) Formen

$$(1.5)\quad \begin{cases} p\{a|\sigma\} = \binom{\sigma}{a}\left(\frac{\kappa\delta}{1+\kappa\delta}\right)^{a}\cdot\left(\frac{1}{1+\kappa\delta}\right)^{\sigma-a} \\ \qquad = \binom{\sigma}{a}\cdot\frac{(\kappa\delta)^{a}}{(1+\kappa\delta)^{\sigma}} \end{cases}.$$

Nu er det en Sjældenhed at Børn i Normalskolen begaar flere Fejl end f.Ex. 8 i en Diktat paa 5[1]–108 Ord (5% i det behandlede Materiale som bestaar af en Skole paa et høiere og en paa et lavt Niveau). Paa mange gunstige Tilfælde kan Binomialloven (1.1) derfor approximeres med Poissons Lov

$$(1.6)\qquad p\{a|\sigma\} = e^{-\mu}\cdot\frac{\mu^{a}}{a!}$$

med Middelværdien

$$(1.7)\qquad \mu = \sigma\varphi(\delta,\kappa),$$

altsaa under Forudsætningen (1.4)

$$(1.8)\qquad \mu = \frac{\sigma\kappa\delta}{1+\frac{\sigma\kappa\delta}{\sigma}} \simeq \sigma\kappa\delta.$$

FIGURE 11. Scanning from the Draft. This figure is a continuation of Figure 10. The text has been translated to English in Appendix A.

mathematical form of the dichotomous Rasch model in the Draft; namely that he was substantiating the appropriateness of the multiplicative Poisson model. Rasch said about this in the Andrich-interview:

> *The discovery of that model* [The dichotomous Rasch model] *actually was an achievement in connection with my work in 1952 in the analysis of the reading tests and the study of the multiplicative Poisson model. I chose the multiplicative Poisson because it seemed a good idea mathematically, if it would work. It turned out that it did work. Then I wanted to have some good motivation for using it and not only the excuse that statistically it worked perfectly. Now I wanted to have a good reason for trying that after I had used it. In order to do so, I tried to imitate the kind of proof of a theorem proved earlier by somebody. I don't remember who. The theorem concerns a number of dichotomous events, each of them having small probability but many of them. Under certain conditions, which can be specified easily including that these probabilities be small, the events becomes Poisson distributed. I tried to imitate that proof and in doing so, I took care that the imitation ended up with just a multiplicative Poisson model, that is to say, there must be a personal factor entering into each of the small probabilities for the dichotomous outcome. The probabilities for the dichotomous case should therefore be of the form* $\frac{\lambda}{1+\lambda}$*, and then the* λ *would have to have a factor that was personal through all, of what we might shall items, and each item, of course, would have a parameter, and then I have my proof.*

Figures 10 through 11 resemblance this proof.

Before accessing Rasch's library, Rasch's statements in the Andrich-Interview about the dichotomous Rasch model's origin as a part of a proof, did not seem important, the point being that not knowing more about the proof, the model could easily have popped up 'at random' in the proof.

From Figure 10 it is seen that Rasch's argumentation for introducing the mathematical form of the dichotomous Rasch model in the Draft is quite identical to the one given in Section 6. This further substantiates that Rasch actually 'discovered' the dichotomous Rasch model as described in Section 6. Furthermore, Figure 10 demonstrates that two stories about the models origin, that is, the one given in Section 6 and the other one about the models origin as a part of a proof, coincide.

In conclusion with regards to Rasch's development of the dichotomous Rasch model, the Draft demonstrates that Rasch wrote down the mathematical form of the model a bit earlier than first anticipated. But, the first time that this mathematical function was used as a statistical model was when Rasch analysed Ravens test (c.f. Page 111).

14 Appendix A. Translations of selected pages of the draft

Translation of the text in Figure 6

Considerations about the distribution of reading sped

Each student achieves a certain level which implies that reading aloud a given text regarded as a stochastic point process – the reading speed varies randomly and the length of each word varies irregularly, the time registered refers to the ending of the reading of the last word in the test, as it is carried out – has a certain average differential reading speed μ: The probability that a word is read through in the time period $(t, t + dt)$ is μdt.

From the theory of stochastic point processes it the follows – what moreover easily is deduced as a borderline case to the law of the binomial – that the number of words σ read during a given time t follows a Poisson law:

$$p(\sigma \mid t) = e^{-\mu t} \frac{(\mu t)^{\sigma}}{\sigma!}$$

and inversely that the time used for reading a given number of word follows the dualistic Γ-distribution:

$$p(t \mid \sigma) = \mu e^{-\mu t} \frac{(\mu t)^{\sigma}}{\sigma!}$$

When comparing two or more tests $(i = 1, ..., k)$ for several students $(\nu = 1, ..., n)$ it is experimentally assumed that μ for the respective tests is proportional with a ξ_ν which states the level of the student's reading ability, while the factors of proportionality α_i – one for each test – characterizes the relative difficulty of the tests:

$$\mu = \mu_i(\xi_\nu) = \alpha_i \xi_\nu$$

Translation of the text in Figure 7 and 8

When the student number ν with the standard ϑ_ν is tested in two dictations D_1 and D_2 with σ_1 and σ_2 words and degrees of difficulty δ_1 and δ_2, then the probabilities

of $a_{\nu 1}$ errors in D_1 and $a_{\nu 2}$ in D_2 will be:

$$p(a_{\nu 1}, a_{\nu 2}) = e^{-(\sigma_1\delta_1+\sigma_2\delta_2)\vartheta_\nu} \frac{(\sigma_1\vartheta_\nu\delta_1)^{a_{\nu 1}}(\sigma_2\vartheta_\nu\delta_2)^{a_{\nu 2}}}{a_{\nu 1}!a_{\nu 2}!}$$

From this it follows that the total number of errors in the two tests

$$a_\nu = a_{\nu 1} + a_{\nu 2}$$

follows a Poisson law (1.6)[3] with

$$\mu = (\sigma_1\delta_1 + \sigma_2\delta_2)\vartheta_\nu, \tag{3.10}$$

and that the probability that a_ν errors is distributed with $a_{\nu 1}$ in D_1 and $a_{\nu 2}$ in D_2 is

$$p(a_{\nu 1} \mid a_\nu) = \binom{a_\nu}{a_{\nu 1}} \left(\frac{\sigma_1\delta_1}{\sigma_1\delta_1 + \sigma_2\delta_2}\right)^{a_{\nu 1}} \left(\frac{\sigma_2\delta_2}{\sigma_1\delta_1 + \sigma_2\delta_2}\right)^{a_{\nu 2}} \tag{3.11}$$

From (3.10) it follows that both tests can be used to estimate ϑ_ν:

$$\vartheta_\nu \simeq \frac{a_\nu}{\sigma_1\delta_1 + \sigma_2\delta_2}$$

provided that δ_1 and δ_2 are known. This result can immediately be generalized to several tests.

An even more important result is evident from (3.11): The distribution of $a_{\nu 1}$ for given a_ν is independent of the parameter, describing the attainment of same the student in question ϑ_ν and the estimate of δ_1/δ_2 can therefore be determined independently of the estimation of ϑ_ν. Furthermore (3.11) can be used as a control of our working hypothesis (1.6)[4] and (1.8).[5]

Translation of the text in Figure 10 and 11

A law of distribution for the number of error in a dictation

A dictation consists of a certain number of words, σ. For simplicity let us assume that all words are equally difficult and whether an error has been committed in the one word does not influence whether an error has been committed in the next.
For a given student it then might be said that there is a certain likelihood, ϖ, for misspelling a word and that the likelihood that a student makes a mistakes in the dictation is given by the law of the binomial:

$$(1.1) \qquad p(a \mid \sigma) = \binom{\sigma}{a} \varpi^a (1-\varpi)^{\sigma-a}$$

[3] (1.6) is $p(a \mid \sigma) = e^{-\mu}\frac{\mu^a}{a!}$
[4] See the previously footnote.
[5] $\mu = \sigma\vartheta\delta$

The likelihood ϖ is given by two factors: the difficulty of a test and the ability of a student to take down a dictation. The degree of difficulty of a test is symbolized by a parameter $\delta > 0$, and the level attained by the student by another parameter $\vartheta > 0$ and ϖ must then be a function of ϑ and δ: $\varpi(\vartheta, \delta)$.
ϑ and δ can be adjusted in various ways. If we arrange it in such a way that $\vartheta \to 0$ the more able the student is and $\vartheta \to \infty$ the poorer the attainment, then we must have

$$\varpi(\vartheta, \delta) \to 0 \quad \text{when} \quad \vartheta \to 0 \tag{3.12}$$

$$\varpi(\vartheta, \delta) \to 1 \quad \text{when} \quad \vartheta \to \infty \tag{3.13}$$

If further $\delta \to 0$ when the test becomes immensely easy and $\delta \to \infty$ when the test becomes immensely difficult, then it seems reasonable to demand

$$\varpi(\vartheta, \delta) \to 0 \quad \text{when} \quad \delta \to 0 \tag{3.14}$$

$$\varpi(\vartheta, \delta) \to 1 \quad \text{when} \quad \delta \to \infty \tag{3.15}$$

The most simple function having these characteristics seems to be

$$\varpi(\vartheta, \delta) = \frac{\vartheta\delta}{1 + \vartheta\delta} \tag{3.16}$$

and accepting this, (1.1) assumes the form

$$p(a \mid \sigma) = \binom{\sigma}{a} \left(\tfrac{\vartheta\delta}{1+\vartheta\delta}\right)^a \left(\tfrac{1}{1+\vartheta\delta}\right)^{\sigma-a} \tag{3.17}$$

$$= \binom{\sigma}{a} \tfrac{(\vartheta\delta)^a}{(1+\vartheta\delta)^{\sigma-a}} \tag{3.18}$$

Now, it is fairly unusual for normal students to commit more then e.g. 8 errors in a dictation which has between 51 and 108 words (5 percent of the observation from a school at a high level and another school at a low level). With very few exceptions, the law of the binomial (1.1) can be approximated by a Poisson law

$$p(a \mid \sigma) = e^{-\mu} \frac{\mu^a}{a!}$$

with the mean

$$\mu = \sigma\varpi(\vartheta, \delta)$$

that is, under the assumption (3.16)

$$\mu = \frac{\sigma\vartheta\delta}{1 + \frac{\sigma\vartheta\delta}{\sigma}} \simeq \sigma\vartheta\delta$$

Georg Rasch. Professor of Statistics at the Faculty of Social Sciences, 1962 to 1971

Lina Wøhlk Olsen[1]

Chapter 4

ABSTRACT This is an account of Georg Rasch as a professor at the University of Copenhagen, Faculty of Social Sciences. Special attention is paid to Rasch's background, the point being that he entered his position wishing to be a professor in another faculty. Rasch's first action as professor, namely to replace the existing curriculum, is also described at some length, as are the very diverse attitudes towards this change. Finally, selected details of Rasch's activities at the Statistical Institute are described to complete the picture of Rasch as an engaged scientist, helpful towards his associates, but at the same time very domineering.

1 Introduction

This paper explores Georg Rasch as a professor of Statistics at the University of Copenhagen, Faculty of Social Science, a position he held from 1962 to 1971.

Rasch was 61 years old when he was appointed professor. At this point in time he had just become internationally recognized for his development of the models that bear his name, namely the Rasch models, or as Rasch chose to refer to them; the Models for Measurement[2]. But, the position was not Rasch's first choice, and consequently, his feelings towards his appointment were not entirely favourable. Furthermore a large segment of the faculty Rasch was to serve did not have him as their first choice either. These preliminary circumstances will be explored in Sections 2 and Section 3.

When Rasch took up his professorship he abandoned the existing curriculum.

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[2]The Models for Measurement and the models that are today referred to as Rasch models are not quite the same. Rasch chose the name, Models for Measurement, to stress the main purpose of these models, namely that they were able to measure. Since then, this class of models have been further developed and are now referred to as Rasch models. But, not all Rasch models adhere to Rasch's definition of a Model for Measurement, that is, not all Rasch models make it possible to obtain measurements in Rasch's definition of the word.

He was to teach statistics to students of economics and students of sociology, but, whereas his curriculum suited the sociologists, the economists saw it as irrelevant. Sections 4 through 8 explore the main differences between Rasch's curriculum and the former one, while Sections 9 through 11 explore the attitudes towards Rasch and his curriculum.

The remaining sections, Sections 12 through 16, describe matters specific to Rasch's activities at the Statistical Institute. We shall see that he had a large scale plan for the Statistical Institute, and that, in general, Rasch was good at getting projects off the ground, but not that successful in finishing them. It will also be demonstrated that Rasch was a committed scientist, who followed those projects that he found interesting, rather than those he felt would be beneficial for the Faculty of Social Sciences. He was in some sense very helpful and caring towards his associates, but at the same time very domineering.

The written sources relevant for this paper are found in annals of the University of Copenhagen, lecture lists, an obituary of Rasch and books about the University of Copenhagen, written in connection with the University's 500th anniversary. As little information about Rasch as a professor of statistics is available in these sources, the main sources of this paper are interviews. I would like to thank Peter Allerup, Ellen Andersen, David Andrich, Michael Weis Bentzon, Ulf Christiansen, Christian Groth, Arne Jensen, Niels-Erik Jensen, Benny Karpatschof, Niels Kærgård, Poul Milhøj, Axel Mossin, Lis Olsen, Jon Stene and Anders Ølgaard for telling me about their memories of Rasch. A brief presentation of their association with Rasch is given in Appendix D. Furthermore, I would like to thank David Andrich for allowing me to use the interviews that he conducted with Rasch. This interview, which will be refered to as the *the Andrich-Interview*, was carried out in the summer of 1979 on the Danish Island of Læsø, where Rasch lived in his retirement.

2 Background

Georg W. Rasch was born in 1901.[3] In his youth he studied mathematics in the University of Copenhagen at the Faculty of Mathematics and Natural Sciences.[4] He was a gifted student, and published his first paper when he was 23 years old, a paper

[3]Section 2 is based on Andersen (1980), Andersen (1986), Hald (1983), Hald (1994), IMSOR (1971), Ramskov (1995), Wright (1980), Årbog (1958-1963) and The Andrich-Interview.

[4]See Chapter 1 of this thesis for further details about Rasch's youth.

written jointly with a professor of mathematics, Niels Nielsen (Nielsen and Rasch, 1923). While Rasch was in his twenties he published six papers on mathematics in various journals, and before he turned 30 he had defended his doctoral thesis (Rasch, 1930).[5]

Rasch's objective in writing his doctoral thesis was to apply for a vacant chair in mathematics, but he was passed over in favour of by the merely 22 years old, Børge Jessen. Rasch felt that Harald Bohr, the professor who had taken interest in Jessen, and later also Jessen himself, thwarted Rasch's opportunities of ever holding a position at the Faculty of Mathematics and Natural Science. Rasch continued, however, to define himself as a mathematician throughout his life.

Beside the chair in mathematics that Jessen got, there were no other positions available in mathematics at the time, and Rasch had to do something to earn a living, particularly considering he also had a wife to take care of. He began to take on minor jobs helping academics from the medical world with the problems they faced when analysing data. During the early 1930's this occupation expanded and Rasch became a statistical consultant. Since Rasch had the backing of some influential people, he was granted two scholarships with a view to getting some formal training in statistics; namely a Carlsberg scholarship for three month's study with the famous econometrician, Ragnar Frisch, and a Rockefeller Scholarship for a one year study with R.A. Fisher, undoubtedly the most famous and influential statistician who ever lived.

Rasch was abroad studying in 1935 and 1936. At this point in time there were only 1 1/3 person employed at the University of Copenhagen to do research in and to lecture on statistics, namely Hans Cl. Nybølle, a professor of statistics at the Faculty of Social Science, and Johann Frederik Steffensen, reader in insurance science at the Faculty of Mathematics and Natural Science. Because Steffesen's time was equally divided between actuarial mathematics, numeric analysis and mathematical statistics, he only counted as 1/3 of a person with regards to doing research in statistics (Hald, 1994, p. 22). The theoretical level of statistics in Denmark was still very limited. Rasch's return from his study leave meant that Fisher's new theoretical ideas on statistics spread to Denmark, in part because Rasch gave lectures at the

[5]The doctoral thesis does not have an equivalent in all countries. In comparison the level of a doctoral thesis is much higher than the level of a PhD thesis.

University of Copenhagen.

From the mid 1930's until Rasch was appointed professor of statistics, his main occupation was to work as a statistical consultant; at first mainly at the Hygienic Institute and at the State Serum Institute. At the State Serum Institute Rasch would usually have a young student working for him. The first of Rasch's students was Anders Hald who started working for Rasch in 1939.

In 1947 Nybølle died, leaving his chair in statistics at the Faculty of Social Science vacant. According to a friend, student and colleague of Rasch, Michael Weis Bentzon, Rasch was not remotely interested in this chair (Int. Bentzon. 27.08.1998). The former student of Rasch, Hald, applied for the chair and was appointed professor of statistics at the Faculty of Social Science, University of Copenhagen in 1948.

In 1950 a committee was formed to discuss the role of statistics at the Universities of Denmark. Rasch was a member of this committee. The committee suggested the establishment of a statistical institute with two departments; one placed in the Faculty of Social Science, and one placed at the Faculty of Mathematics and Natural Sciences. Each department was to be governed by a professor of statistics. As to the question of whether Rasch would be a natural choice as professor of statistics at the Faculty of Mathematics and Natural Science, Bentzon, says (Int. Bentzon. 27.08.1998):[6]

> *Yes! And Hald thought so too then. What I remember is that Hald tried to make Rasch write something. He had not written anything. His contributions in consulting did not qualify him for a chair, and I think that Hald did all he could to make Rasch write something. But Rasch did not do that until 1960 when he published the Models for Measurements. And that was the first substantial publication from Rasch's hand. There were some articles, of course.*

Arne Jensen, who became a professor of statistics at the Polytechnical College in 1963, but who more importantly was also a student, colleague and friend of Rasch says (Int. Arne Jensen, 08.02.2001):[7]

> *The mathematicians wanted Hald and not Rasch. But we all agreed that it*

[6]The text has been translated from Danish. The original text is given in Appendix E.
[7]The text has been translated from Danish. The original text is given in Appendix E.

Rasch (left) in conversation with Hald (middle) and Hald's wife (right). The picture was taken when professor Erling B. Andersen graduated as the first cand.stat. in 1963 (cand.stat is approximately the same as master of science in statistics).

> *was Rasch who had initiated everything, and that it was he who ought to be professor.*

In the end the Faculty of Mathematics and Natural Sciences did not proceed with the plan to have a department of statistics at their faculty at this point in time, and it was therefore decided to establish a statistical institute under the Faculty of Social Science with only one department. The institute was named 'The Statistical Institute'[8], and it was established in 1953. Rasch said in the Andrich-Interview:

> *Well, I think the question of getting a properly founded statistical institute was raised by Harald Bohr, the brother to Niels Bohr. He was a mathematician and a very good one at that. He had already raised that question around 1950. But of course, it took some time. Meanwhile Hald had become a professor and was set to be the first head of that institute. There was, clearly to me, an arrangement in the Science Faculty, in which Hald, probably against his will became involved.*

In 1958, after some five years, the professors of mathematics, B. Jessen, W. Simonsen, W. Fenchel, T. Bang and Einar Andersen submitted an application to the Ministry

[8]In Danish: Statistisk Institut

of Education and Research concerning the establishment of a statistical institute at the Faculty of Mathematics and Natural Science. Two years later, in 1960, a chair in statistics was established. However, Rasch was not given the opportunity to apply for this professorship. The professors of mathematics had recommended that the position *not* be advertised, but that Hald was appointed professor without notice (Årbog, 1958-63. II. p. 231 ff.). According to Arne Jensen, Hald had repeatedly turned down the suggestion by the mathematicians that he should be a professor in the Faculty of Mathematics and Natural Sciences. In this way, Hald had done his utmost to promote his former teacher and mentor to the chair. However, as the mathematicians, Bohr and Jessen, could not accept Rasch as professor, in the end, Hald gave in (Int. Arne Jensen, 08.02.2001). Rasch said in the Andrich-Interview:

> *I was told by a friend in the Faculty of Science who was in fact sent to give me a message, that an institute in mathematical statistics there was going to be established but that the headship wouldn't be given to Dr. Rasch. It would be given to his student, the then already Professor Hald. That was a frightful blow to me. I think not much should be said about that part [...]*
>
> *My old friend Tranekjær Rasmussen*[9] *whom I knew from 1919, became aware that I had not applied as yet for the position vacated by Hald. Then he got busy and told me that I had to apply for it even if I didn't want to have it. He said I had to show my competence because others in the Faculty of Sciences were saying that I had never published anything. Now he knew that I had published a lot of things, and very important things. He insisted I should not let them get the wrong impression. He indicated I should apply and send in the necessary documentation for the application.*

3 The appointment

Rasch applied for the chair in statistics at the Faculty of Social Science. So did Gustav Henning Leunbach and Ernst Lykke Jensen.[10]

The role of statistics at the Faculty of Social Sciences was partly to be an aux-

[9] Rasch and Tranekjær Rasmussen studied mathematics at the University of Copenhagen at approximately the same time. They were both students of Niels Erik Nørlund, who was professor of mathematics. Like Rasch, Tranekjær Rasmussen did not get a position in mathematics, and he turned to psychology. He was appointed professor of psychology at the University of Copenhagen in 1939.

[10] Section 3 is based on Hald (1983), Hald (1994), Årbog (1958-1963).

iliary science. It was intended that the Statistical Institute would give lectures in statistics for students of economics and sociology, and that the scientists within the faculty, that is, economists and sociologists, would receive statistical advice should such help be needed. Amongst these two groups, the economists were by far the largest segment of people.

Amongst the three applicants, Rasch, Leunbach and Lykke Jensen, the teachers in economics preferred Lykke Jensen as Hald's successor (Int. Christiansen, 12.02.2001), (Int. Ølgaard, 15.11.2000). Lykke Jensen was a student of Hald, and he had worked at the Statistical Institute since 1954; in the beginning as research assistant, later as an associate professor.[11] Working at the Statistical Institute, he had been teaching since 1957, and was therefore well known by the economists, especially so because he was an economist by training. Anders Ølgaard, who was appointed acting professor[12] in 1962 at the Institute of Economics, Faculty of Social Sciences, and appointed professor in 1966, says (Int. Ølgaard, 15.11.2000):[13]

> *Lykke Jensen was, to be sure, an economist, and he concerned himself with those statistical methods etc, which were prevalent at the time, and which have largely remained standard within economic research and econometrics which to say the least, Rasch had not done.*

When Rasch applied for the chair he was an associate professor[14] in biological statistics at the University of Copenhagen. His main occupation was, however, still as a statistical consultant; mostly at The Danish Institute of Educational Research (DPI) and The Military Psychology Group (MPA). In connection with his work on educational statistics Rasch had developed the Models for Measurements or the Rasch models, which he had recently published (Rasch, 1960), (Rasch, 1961). When he applied for the chair in statistics, he sent in a list of his most substantial papers, and this list included 86 papers and books. The other two applicants, who were some 30 (Lykke Jensen) and 20 (Leunbach) years younger, did not have such an impressive list of papers. On the other hand, amongst the applicants, only Lykke Jensen had experience in economics.

The members of the appointment committee were respectively, P. Nørregaard Ras-

[11] In Danish: amanuensis.
[12] In Danish: konstitueret.
[13] The text has been translated from Danish. The original text is given in Appendix E.
[14] In Danish: lektor.

mussen from Copenhagen, Leo Törnquist from Helsingfors and Herman Wold from Uppsala. Both Törnquist and Wold were professors of statistics, while Nørregaard Rasmussen was a professor of Economics. Nørregaard Rasmussen was a strong figure in the Institute of Economics, Faculty of Social Sciences in 1962.

The committee agreed that Leunbach was not qualified, but with regard to Lykke Jensen, the opinions were divided. While Törnquist and Wold judged Lykke Jensen unqualified for the chair, Nørregaard Rasmussen argued that Lykke Jensen had other qualities that ought to be taken into account. In the committee's recommendation this is stated as follows (Årbog, 1958-63, p. 75):[15]

> *A minority, P. Nørregaard Rasmussen, is of the opinion that Lykke Jensen has proved himself qualified, and the minority emphasizes that the advertised chair in statistics comes under the jurisdiction of the Faculty of Social Science. The teaching is aimed at students of economics, and the supervision and the cooperation within the faculty will be with and for economists. With reference to this, the minority emphasizes that not only is Lykke Jensen an economist by training, but in addition, in all his papers he has shown an interest in a number of statistical methods of importance in economics. Furthermore, in applying statistical methods he has, almost without exception, turned towards economics.*

There is one event that occured at the time the appointment committee was in Copenhagen that must be mentioned, in part because it certainly did not improve Rasch's reputation among the economists, in part because it illustrates two traits of Rasch, which might not be polite to mention, but which nonetheless were integral parts of his character, namely that Rasch could be rude to people he didn't get on with and that he occasionally got intoxicated.

The event has to do with a lecture Wold gave on econometrics. Rasch and Wold knew of each other, and had been to some of the same conferences (Rasch, 1953), but Rasch did not approve of Wold's ideas on econometrics (Int. Christiansen, 12.02.2001). When Wold was giving the seminar, Rasch turned up half way through and interrupted Wold by saying something like "Herman Wold, you have not understood anything" (Int. Stene, 08.10.2002). This event left a somewhat strong impression on the economists who attended Wold's lecture, especially so because Rasch was intoxicated. Ølgaard says (Int. Ølgaard, 15.11.2000):[16]

[15]The text has been translated from Danish. The original text is given in Appendix E.
[16]The text has been translated from Danish. The original text is given in Appendix E.

> *And it made a strong impression on the economists. Not because we never drank alcohol, but we did not even know the man, and we thought: "that was quite peculiar!". But what I am saying with this is that he was totally different from the rest of us. And what happened next was that he was appointed, but it is safe to say that he never really got on the same wavelength as most of us. And I might even add that he was not interested in doing so.*

Rasch was a passionate man. His support and friendship was whole-hearted, but when he disagreed with someone or something, he was equally unreserved in his criticism, totally disregarding the consequences (Int. Karpatschof. 13.01.2003). In this case it was a professorship that was at stake. But, Rasch strongly felt that econometrics was used incorrectly, and he was very outspoken about it. Jon Stene, who was a colleague of Rasch at the DPI from 1963, and affiliated with the Statistical Institute from 1964 says about the unfortunate event at Wold's lecture (Int. Stene, 08.10.2002):[17]

> *And it was, shall we say, the beginning of a conflict which just continued. Rasch said that those economists did not know what they were doing. He thought they should use methods other than regression analysis.*

Axel Mossin, who was a student of economics when Rasch was appointed as a professor, and who later joined the statistical Institute, emphasizes that the event portrays a characteristic of Rasch's personality. He says (Int. Mossin, 24.09.2002):[18]

> *But is was also a manifestation of Rasch's level; he did not cringe before anybody.*

In spite of Rasch's harsh statement, Wold voted for Rasch when it came to who should be professor. The appointment committee concluded in unison that Rasch was better qualified than Lykke Jensen, and Rasch was appointed professor of statistics at the Faculty of Social Science from the first of June, 1962, though not at the faculty he had hoped for, and not at a faculty that had Rasch as their first choice either.

[17]The text has been translated from Danish. The original text is given in Appendix E.
[18]The text has been translated from Danish. The original text is given in Appendix E.

4 The curriculum of Rasch's predecessor

Rasch's predecessor to the chair in statistics, Hald, had introduced two lines of statistics for the students to choose between, namely the course named Advanced Statistics[19], running over three semesters, and the course named Short Statistics,[20] running over one semester.[21]

Most of the students attended the course, Short Statistics. The text book used for this course was specifically aimed at introductory courses in statistics for students of business, public administration or economics, and the mathematical level of the book was moderate. Nonetheless, most standard methods were introduced (Neter and Wasserman, 1961).

The course, Advanced Statistics, was much more demanding than the course Short Statistics, and the students who attended this course learned quite a lot about the modern theories of the time. The textbook for his course was written by Hald. Professor Ellen Andersen, who in 1965 got a position at the Institute of Economics, and in 1973 was appointed professor of Empirical Economics, attended Advanced Statistics as a student. She has told about what it was like to be a student of Hald (E-mail. Ellen Andersen, 29.08.2002):[22]

> *Hald's students knew analysis of variance and regression analysis by heart, and there were multitudes of economic data in the exercises, e.g. from a thick booklet containing data from a number of farms. In a piece of home-work I estimated the demand functions for various fruits and vegetables, based on the market gardeners' sales.* [...] *I felt well-equipped with statistical theory and methods of econometrics,* [...]

5 Taking over the Statistical Institute

When Rasch took up his professorship, he decided to abandon the former curriculum. The former courses in statistics, Advanced Statistics and Short Statistics, were gradually phased out, and all new students had to attend Rasch's 'new statistics',

[19] In Danish: Stor Statistik.

[20] In Danish: Lille statistik.

[21] Section 4 is based on Davidsen (2001), Hald (1952), Neter and Wasserman (1961), E-mail. Ellen Andersen (29.08.2002), Int. Christiansen (12.02.2001) and Int. Niels-Erik Jensen (27.06.2001).

[22] The text has been translated from Danish. The original text is given in Appendix E.

a course that ran for two semesters, starting from the fall of 1962.[23]

Rasch employed two teaching assistants,[24] namely Ulf Christiansen and Poul Christian Matthiessen, to assist him with the practical details of writing the new textbook. One of the preliminary tasks was to find data to illustrate the statistical theory. Rasch had a lot of data that served nicely to illustrate some of the statistical models he planned to teach, namely data from various attainment tests, but with regards to examples that were relevant for students of the social sciences, he had next to no data. Matthiessen worked at the Danish Department of Statistics[25] (today: Statistics Denmark[26]), and in this capacity he knew a lot about what kinds of data were available. In practice, this implied that Matthiessen provided data for a lot of the examples in the textbook, Rasch (1966). But, as opposed to Hald's courses, examples from economics were a rarity (Rasch, 1966).

The new textbook was finished in 1963 (IMSOR, 1971). Christiansen, who took notes while Rasch lectured, wrote it, which implies that the first year of students did not have any textbook. The intention was that the students should continually get Christiansen's notes. However, as Mossin, who as a student of economics attended Rasch's course, says (Int. Mossin, 24.09.2002):[27]

> *And you have probably heard that it progressed in a sort of chaotic way, because, you see, Rasch had this idea that he would revolutionize the course in statistics. And there were no textbooks he saw fit for the purpose.* [...] *I dare say the plan was that this* [that the students should continually get Christiansen's notes] *should proceed continually, but gradually, more and more time elapsed from the time of the actual lectures to the time the written notes were available.*

During 1963, the transcribed lecture notes were written up, and then collected as a set of typewritten papers. In 1966 this set of notes was published as a book titled *Account of Professor G. Rasch's Lectures on the Theory of Statistics*.[28] A brief outline of the contents of the book is presented in Appendix A.

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[23]Section 5 is based on Davidsen (2001), Wolf (2001), Int. Christiansen (12.02.2001), Int. Kærgård (26.02.2001), Int. Stene (08.10.2002).

[24]In Danish: Undervisningsassistenter.

[25]In Danish: Statistisk departement.

[26]In Danish: Danmarks statistik

[27]The text has been translated from Danish. The original text is given in Appendix E.

[28]In Danish: *Referat af Professor G. Rasch's forelæsninger over statistikkens teori.*

Few students approved of the change from Hald's courses to Rasch's course in statistics. Christiansen says about this (Int. Christiansen, 12.02.2001):[29]

> *The former course in statistics had been divided into Short and Advanced statistics, where Hald lectured for those who attended Advanced Statistics.* [...] *And then there was Short statistics, which was run by E. Lykke Jensen. And it was this course that most of the students attended.*[...] *But then Rasch came along, and then everybody was exposed to something, which they by no means considered as easy. Of course, there was some who thought it was fine, but that was certainly a minority.*

The reason that Rasch's course was considered difficult was the level of mathematical sophistication. Mossin says (Int. Mossin, 24.09.2002):[30]

> *Statistics was considered to be tremendously hard; the use of mathematics to that extent. The training in economics was also less formalized at that time. And even something with a small degree of formalization was considered to be extremely difficult by the students.*

Poul Milhøj, a part-time teacher[31] at the Institute of Economics in 1962, and later a Professor at the Copenhagen Business school says (Int. Milhøj, 04.07.2002):[32]

> *There was a clear line from Nybølle to Hald to Rasch, that it became more and more theoretical.*

Rasch was aware that the basic course in statistics would be considered difficult, and from the very beginning he hired four teaching assistants[33] with a view to having class room exercises following up on the lectures (Int. Christiansen, 12.02.2001). However, in general Rasch was not too concerned about the fact that few students understood his lectures. Stene says about this (Int. Stene.29.10.2002):[34]

> *Rasch was indeed a professor from the old school. He said that "If five percent of the students understand what I am saying, then I am happy."*

The fact that most students thought that the new curriculum was too difficult, was probably a minor problem compared to the economists' attitude toward the contents of Rasch's curriculum. This attitude can be illustrated by the following quote from

[29]The text has been translated from Danish. The original text is given in Appendix E.
[30]The text has been translated from Danish. The original text is given in Appendix E.
[31]In Danish: Ekstern Lektor.
[32]The text has been translated from Danish. The original text is given in Appendix E.
[33]In Danish: Undervisningsassistenter
[34]The text has been translated from Danish. The original text is given in Appendix E.

a book about the University of Copenhagen, Faculty of Social Science (Kærgård, 2001, p. 177):[35]

> *Rasch was internationally recognized and a most resourceful statistician. But his ideas and methods fitted badly into what was standard amongst economists. He looked upon the commonly used methods based on the normal distribution, eg. regression analysis, with the uttermost criticism. He did away with the old choice between Advanced and Short statistics, and introduced his own new statistics. In his time there was a substantial distance between what the Statistical Institute was concerned with and what the econometricians at the Institute of Economics worked with.*

Seemingly, the sociologists were quite content with Rasch most of the time.

Today, it might appear rather drastic for a new professor to simply abandon the former curriculum, as well as the former way of teaching. However, the professors of the 1960's were not placed under such restrictions as professors are today. The professors reigned as kings within their domains, and Rasch's actions were not unprecedented at that time.

When Rasch took over the Statistical Institute, Lykke Jensen still held a position there, as did Rasch's friend and former student, Arne Jensen, who held a position as an external lecturer.[36] Both were appointed professors soon after, namely in 1962 and 1963 respectively, implying that Rasch was free to choose whatever staff members he wanted. As already mentioned, Rasch employed Matthiessen and Christiansen as research assistants, using some of the available permanent positions to do so. As the years went by, the staff changed but all in all, the Institute grew in size, as did most of the University over that period. A table of the permanent positions available at the institute may be found in Appendix B.

[35]The text has been translated from Danish. The original text is given in Appendix E.
[36]In Danish: ekstern lektor.

6 The Models for Measurement

As mentioned, Rasch wanted to revolutionize the course in statistics.[37] In the curriculum the major change was the introduction of Rasch's own 'invention', namely the Models for Measurement. Rasch had developed these models as a result of his consulting work for psychologists and educationalist on the analysis of intelligence tests, and he had recently published his discoveries in the book Rasch (1960), and the article, Rasch (1961). The Models for Measurements were especially aimed at analysing questionnaires and attainment tests, but, while such kind of data were often analysed in sociology, they were seldom used in economics. Nonetheless, Rasch's exposition of these models formed a large part of his first textbook,[38] Rasch (1966). In the Andrich-Interview Rasch says about his initial motive for introducing these models in the curriculum:

> *I asked myself what I should teach to students of the Social Sciences, that is, sociologists and economist, in fields in which I had no experience to speak of. I realized that what I had written in the book in 1960* [*(Rasch, 1960)*] *would not suffice, not at all.* [...] *However, I saw that here was a tool that could be worked out to deal with problems they do meet in sociology and economics. Then I decided that if I could teach a large group of models and use could be made of it, then I would have done something that is useful to the social sciences.*

It is well documented that Rasch really thought that the Models for Measurement were important (Rasch, 1977), and not just as an application within educational statistics and psychology. Erling B. Andersen, to whom Rasch gave private lessons on the Models for Measurement in the early 1960s, and who succeeded Rasch as the chair in statistics, writes about the name Rasch chose for these models (Andersen, 1995, p. 384):

> *. . . he was very eager not to call the model the 'RM'* [*The dichotomous Rasch model*]*. Instead, he suggested the name 'models for measurement'. I felt at the time that it was an awkward name and sounded too much like 'measurement models', which was commonly used for quite different types of models. At first I thought it was a trait of modesty in Georg Rasch's*

[37] Section 6 is based on Andersen (1980), Andersen (1986), Rasch (1966), Rasch (1968'), Rasch (1968), Rasch (1969), Rasch (1972), Int. Allerup (26.02.1998), Int. Christiansen (12.02.2001), Int. Kærgård (26.02.2001), Int. Mossin (24.09.2002), Int. Stene (08.10.2002).

[38] See Appendix A for an overview of the contents of the first textbook.

personality. But, on second thought I did not find this trait of modesty consistent with my general impression of his personality. In subsequent discussions I became more and more convinced that it was not modesty, but that his suggested name had the clear purpose of stressing the most important property of the model: That it solved a basic measurement problem in the social sciences, or, as became clearer later, in Georg Rasch's opinion, in all sciences.

Rasch's basic idea was that the Models for Measurement made it possible to *measure properly*, and, equally importantly, to validate which data conformed to measurement and which did not: Rasch had specified demands for a social sciences measurement to be of the same quality as measurements in the natural sciences and he had then found out exactly which kind of statistical models conformed to these specified requirements, namely the Models for Measurement (Rasch, 1968). The conclusion therefore was that a given data set yielded measurements in Rasch's well-defined meaning of the word, if, and only if, the data conformed to one of the Models for Measurement. So, if the Models for Measurement did not describe the data, then, in certain situations, it was considered better to discard the data than the model. This view of Rasch's upon data was indeed controversial and quite a contrast to the traditional approach where the statistical model was expanded to fit the data.

Closely connected to the Models for Measurement was the concept of Specific objectivity, which by and large was the name Rasch chose for his requirements for measurements. Rasch used a lot of energy during the 1960's and the 1970's trying to convince people of the need for specific objectivity in their analysis, and hence the need to use the Models for Measurements. Actually, he did this so thoroughly that he was recognized as some sort of missionary (Int. Christiansen, 12.02.2001), (Int. Kærgård, 26.02.2001). Stene says (Int. Stene, 08.10.2002):[39]

The trait that characterized Rasch's manner was downright preaching - Evangelical - The fundamentalist style, which comes from Evangelism. It was quite the same structure as a gospel reading. [...] *And also this thing about thrusting oneself on others, and trying to get them into your community.*

[39]The text has been translated from Danish. The original text is given in Appendix E.

7 Empirical approach and tailored methods

An aspect of Rasch's revolution of the curriculum was that he tried to link together empiricism and statistical modelling.[40] As a means of achieving this, a substantial part of the lectures went into describing a large-scale coin tossing experiment. In doing so Rasch tried to communicate his general view of statistics, namely that data was generated by some chance mechanism or other, and that the goal was to uncover this mechanism. Mossin says about attending his lectures (Int. Mossin. 24.09.2002):[41]

> *And, many of the first weeks went into describing this large scale coin tossing experiment, and that is certainly a sort of empirical approach to statistics.* [...] *It might be the case that those long introductory examples were over the heads of most students.*

In line with Rasch's emphasis on empiricism was his emphasis on the examples in his textbooks. This can be illustrated by the following quote from Christian Groth, who today is an economist and holds a position as associate professor at the Institute of Economics, but who in the mid 1960s was a student of sociology, and as such attended Rasch's course in statistics (Int. Groth, 19.04.2002):[42]

> *He showed deference for data. Data was something to be taken seriously. And data should be handled thoughtfully, and not just according to the rules. You should not just apply ready-made methods, because it might be the case that the data said something different.* [...] *A point of view which I remember him having and which I thought was quite right, was that any empirical analysis was not merely a quantitative statistical analysis. It was also a case study, where you tried to find out what kind of causal effects were present. The statistical analysis ought to be combined with something like that, that is, with the appropriate branch of science. It was important that the statistician worked together with a scientist from that branch of sciences.*

Hence, Rasch did not 'merely' teach statistical theory. Using the examples, he taught about the interplay between data, statistical modeling and selected branches of the sciences.

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[40] Section 7 is based on Andersen (1980), Rasch (1966), Rasch (1968), Rasch (1969), Int. Christiansen (12.02.2001), Int. Groth (19.04.2002), Int. Niels-Erik Jensen (27.06.2001), Int. Mossin (24.09.2002), Int. Stene (08.10.2002)

[41] The text has been translated from Danish. The original text is given in Appendix E.

[42] The text has been translated from Danish. The original text is given in Appendix E.

Seemingly, handling data thoughtfully and discarding data which did not fit one of the Models for Measurement are two very different thing – but only seemingly. The Models for Measurements are chiefly used to analyse questionnaires, and they are a tool by which it is possible to validate whether a questionnaire makes sense. That is, instead of just applying a statistical model that fitted the data, Rasch would find out, preferably in cooperation with an educationalist or a psychologist, which trait the questionnaire was supposed to measure. Then, by applying a model for Measurement, he would validate the success of the questionnaire to measure this trait. If some questions of a questionnaire did not fit, these questions would not *just* be discarded. Rasch would investigate *why* they did not fit. So, also in this case, Rasch handled data very thoughtfully.

Stene points out that Rasch was quite content if five percent of the students understood his points, and there were certainly quite a few of the students who did not. Therefore, it is generally believed that Rasch failed to communicate his point of view on the interplay between statistical modeling and data to the students.

8 Dislike of 'standard methods'

The counterpart of Rasch's emphasis on tailored methods was his dislike of standard methods.[43] The standard methods of that time were, more or less, all based on the normal distribution, such as analysis of variance, analysis of regression and factor analysis. There is every indication that Rasch thought that such methods were used carelessly and unscientifically. The following quote from Rasch's first textbook substantiates this (Rasch, 1966, A.18. p. 1):[44]

> *Formerly it was universal dogma that distributions, except for those that were 'born' discontinuous (as e.g. coin tossing experiments), ought to be normal distributions.* [...] *Whether the normal distribution fitted the observations was seldom investigated.* [...] *Nonetheless, numerous tests based on normality were carried out as a test of other specified hypotheses. This approach is, of course, only permissible when the observations or some transformation of them can be described by the Gaussian law. But this will*

[43] Section 8 builds on Rasch (1966), Rasch (1968), Rasch (1969), Int. Allerup (26.02.1998), Int. Christiansen (12.02.2001), Int. Groth (19.04.2002), Int. Niels-Erik Jensen (27.06.2001), Int. Mossin (24.09.2002), Int. Stene (08.10.2002).

[44] The text has been translated from Danish. The original text is given in Appendix E.

> *quite often reveal itself not to be the case, and, if it is not, even minor deviations will impair the basis for decisions for a number of commonly used statistical tests.*

It is quite appropriate for a professor of statistics to warn of the dangers of or discourage the use of ill placed methods. However, the problem was that Rasch preferred *not* to teach the standard methods. For instance, regression analysis, which was commonly used in economics, did not enter into his first textbook, Rasch (1966), and Rasch's exposition of the normal distribution was very limited. Niels Kærgård, who was a in the 1960s student of economics and teaching assistant at the Statistical Institute, and today is a professor of Agricultural Economics, says about the implications of Rasch's attitudes (Int. Kærgård, 26.02.2001):[45]

> *So in some sense or other I think you can safely say that it was rather catastrophic for the relations between economists and statisticians because what the economist actually used was analysis of regression and the normal distribution. And Rasch was extremely critical of both and did away with both of them.*

Actually, Rasch did teach the normal distribution. Maybe not from the beginning in 1962, but quite soon thereafter a booklet that dealt with the normal distribution and elementary analysis of variance was included in Rasch's curriculum.[46] There is, however, no doubt that Rasch preferred to teach his students to be cautious of the normal distribution rather than teach them how to use it.

It seems as if Rasch's choice not to teach the standard methods based on the normal distribution was but one part of the problem. Another part was that he was so extreme in his criticisms. We have already seen how Rasch criticized Wold, a member of his appointment committee, at a seminar (c.f. Page 140). And such extreme outbursts were not one-off events (Int. Allerup, 26.02.1998). The following quote shows yet an example (Andersen, 1995, p. 385):

> *At certain occasions, when we had all consumed a generous amount of alcohol, he would invite all persons present to a party on his front lawn to burn all books containing the word 'normal distribution'.*

[45] The text has been translated from Danish. The original text is given in Appendix E.

[46] This booklet was the one Rasch had used as a textbook when he gave lectures in elementary statistics to students of psychology, and though it was written by F. Abildgård Jørgensen, it was Rasch who was responsible for the contents of the book (Jørgensen, 1957).

Such obstinacy was of course a problem, especially because the economists were equally convinced that the normal distribution was one of the most important tools in statistics.

9 The students

Up until the early 1960s, the prevailing principle in academia was professorial power.[47] However, these old ways were on the point of extinction. Furthermore, the number of students exploded, as the baby boom of the late 1940's was catching up on the university system.

Stene remembers that when he first started teaching in 1964 at the University of Copenhagen, the students were quite meek. But, over the next couples of years this changed, and the students began to voice their opinions. Some of them were dissatisfied with Rasch's way of teaching and with his teaching material. On perhaps half a dozen occasions, a small group of students approached Rasch in his office to complain about the contents as well as the comprehensiveness of his textbook. Each time Rasch got quite enraged. First of all, being used to professorial power, Rasch was infuriated with the very idea of students complaining. Second, as far as their complaints were concerned, he thought it to be pure nonsense for students to assume to know what they wanted to be taught, since, in his opinion, they knew too little about statistics to know what was best for them. Furthermore, Rasch did not care much about their complaints about the lack of comprehensiveness of his lectures and the textbook. He was quite content if five percent of the students understood his teaching, and, as to the rest, he thought them too stupid to be taken into consideration. As Stene says, "Rasch was quite the elitist" (Int. Stene, 29.10.2002).

It is hard to tell exactly how good or how bad Rasch's lectures were. Presumably, their quality varied over the years. As it happened, in the years preceding Rasch's appointment, Rasch's staff members, especially Stene, made supplementary notes to the first textbook, Rasch (1966). Whats more Rasch delegated much to his associates while he mainly lectured on the topics he found interesting. This calmed down the students' unrest.

On the other hand, Peter Allerup, who worked at the Statistical Institute in the late 1960s, and who has continued to work with the Models for Measurement at the

[47] Section 9 builds on Davidsen (2001), Wolf (2001), Int. Allerup (26.02.1998), Int. Kærgård (26.02.2001), Int. Stene (08.10.2002).

Danish University of Educational Research (formerly the Danish Institute of Educational Research), where he now holds a position as professor, admits that in the late 1960s, quite often Rasch's lectures were not that good, and that the students would leave. Actually, nobody wanted to be amongst the last to leave, because if there were only a few left, Rasch would venture to sit amongst the remaining students, giving them lectures face to face. On the other hand, Allerup remembers that Rasch's lectures on topics he cared about were very inspiring; in part because Rasch was so committed himself. Allerup therefore concludes that Rasch gave good lectures when he was interested in the topic, but that Rasch, at least in the late 1960s, found it tiresome to teach the basic stuff over and over again (Int. Allerup, 26.02.1998).

Even though Rasch dismissed every complaint the students voiced, this must not be construed to mean that he preferred the students to be silent. It was quite all right for a clever student to ask questions. If a clever student showed an interest in statistics, Rasch could be very helpful (Int. Stene, 29.10.2002).

Of course, there were also students who enjoyed Rasch's lectures. For instance, Groth, who attended Rasch's lectures in the mid 1960s, says (Int. Groth, 19.04.2002):[48]

> *He also had this habit of thinking while he taught. He did not just reel off a speech. It was almost as if he did research while he gave tutorials. Of course, there were many who gave up on him, and thought him to be an inferior teacher, which he might have been in some sense. But, on the other hand, it was very interesting.*

Furthermore, it seems that if the students understood Rasch's points, then they benefited from his lectures, particularly with respect to his idea about trying to find out what the data really was about. Mossin, who attended Rasch's lectures in the beginning of the 1960s, says (Int. Mossin, 24.09.2002):[49]

> *In my opinion, he was tremendously inspiring, that is, the empirical approach, where you really have to dig into the data before you use any statistical models.* [...] *He was also eccentric, absolutely. He had this completely overrated view on the importance of the Models for Measurement.*[50]

[48]The text has been translated from Danish. The original text is given in Appendix E.

[49]The text has been translated from Danish. The original text is given in Appendix E.

[50]For those who know about Niels Bohr and his principle of complementarity: Rasch was said to overstate the applicability of the Models for Measurements in exactly the same way as Niels Bohr overstated the applicability of the principle of complementarity (Int. Mossin, 24.09.2002).

10 The economists

The function of the Statistical Institute was to give tutorials in statistics to students of economics and in sociology and to provide help in Statistics to the academics in the Faculty of Social Sciences.[51] However, the economists and the sociologist had very different points of view on Rasch and his curriculum. In this section, the attitude of the economists will be explored, while the following section will deal with that of the sociologists.

As branches of sciences at the University of Copenhagen, Economics and Statistics had parted from political science, into a so-called 'Economics-Statistical Laboratory'. In 1953, with the establishment of the Statistical Institute, statistics parted from this union, and five years later, in 1958, the Institute of Economics was founded. As mentioned, it was Hald who established the Statistical Institute, and while Hald held the chair the economists had relatively few complaints. However, once Rasch became a professor, things got much worse for the economists. Ellen Andersen writes (E-Mail. Ellen Andersen, 29.08.2002):[52]

> *Rasch began his tutorials in statistics by abolishing the existing curriculum, and in the first academic year he gave lessons without any teaching material, which was reckless. He eliminated analysis of variance and analysis of regression and put in its place his own method where each data set had it's own tailed statistical model, which was fine, but not easily imitated. Hector Estrup, who was the first to give tutorials in Johnston's* Econometric Methods *was not able to skip the introductory chapters on regression analysis. So, statistics and econometrics drifted apart.*

One of the reasons that Rasch's tailored methods were especially difficult to imitate for the students of economics was that there were very few examples from economics in Rasch's course. If Rasch had given some examples of how to analyse economic data, the students would probably have been able to imitate and further develop whatever tailored method Rasch would have developed. Groth says about this (Int. Groth, 19.04.2002):[53]

> *There were very few examples from economics. So, his examples were not*

[51] Section 10 is based on Davidsen (2001), Int. Niels-Erik Jensen (27.06.2001), Int. Kærgård (26.02.2001), Int. Ølgaard (15.11.2000).

[52] The text has been translated from Danish. The original text is given in Appendix E.

[53] The text has been translated from Danish. The original text is given in Appendix E.

> *directed towards the economists. I think that was largely why the students of economics did not think it was so interesting.* [...] *I am sure that much of what we learned in Theoretical Statistics has been quite useful to me as an economist, stochastic processes and the like. But, it was not economics and economic data that he and his enthusiastic colleagues worked with. You could feel that. This issue about the examples was very important.*

Consequently, the state of affairs for most of the students of economics was that they did not see statistics as important. Kærgård, who, as a student of economics attended Rasch lectures, says (Int. Kærgård, 26.02.2001):[54]

> *We learnt something quite different to what we, in some sense or other, thought we needed.*

Furthermore, since Rasch was of the opinion that the economists used regression analysis in a wrong and unscientific way, and since he made this viewpoint abundantly clear, it is not surprising that the teachers in economics also distanced themselves from him. Ølgaard says (Int. Ølgaard, 15.11.2000):[55]

> *So Rasch had to acknowledge that we thought that the traditional methods of econometrics and statistical approaches (which are used today I might add), that these were those that were important. And it might be that Rasch was partly right in his criticism, but, the methods that he offered, and which I am not able to judge critically were actually not interesting.*

As Rasch's curriculum drifted away from what the economists were doing, his communication with the economists became sparse. Ølgaard says (Int. Ølgaard, 15.11.2000):[56]

> *Talking about Rasch and the rest of us – and I was appointed in 1962 so I have seen it all – then the interesting thing is that we had so very little contact and we scarcely knew, how Rasch looked at it. But it was more or less implied that those interested in econometrics and statistical theory as standardly conceived, lost contact with Rasch.*

Ellen Andersen further explains (E-mail. Ellen Andersen, 29.08.2002):[57]

> *But there were no personal disagreements – just no cooperation. We tried, though; Rasch and I had a couple of meetings, I think, where he plotted*

[54]The text has been translated from Danish. The original text is given in Appendix E.
[55]The text has been translated from Danish. The original text is given in Appendix E.
[56]The text has been translated from Danish. The original text is given in Appendix E.
[57]The text has been translated from Danish. The original text is given in Appendix E.

> *some time series on his graph paper,* [...] *while I explained about the work with ADAM.*[58] *I remember in particular that I explained what a demand function was, and why I assumed that demand was determined by income. Rasch was not with me there; He earned the money he needed and not the other way around. The problem was that he had his, random and data based, model and I had my economic theory and econometric tradition based recipe for a macro model, and we did not really have any need of one another.*

In effect, the economists went about their own business, which at this point in time became more and more specialized. Formerly, most chairs in economics had no other specification than that the chair was in economics. This was changing in the 1960s and 1970s. Of interest here is empirical economics. Ellen Andersen was the prominent figure in its development. She began working at the Institute of Economics in 1965, and in 1971, Ellen Andersen, together with three young PhD students,[59] Niels Kærgård, Christian Hjorth-Andersen and Rolf Norstrand, moved to new offices, which soon became a department of empirical and applied economics. In 1973 Ellen Andersen was appointed professor of empirical economics. In this way, the development of empirical and applied economics happened entirely independently of Rasch.

11 The sociologist

In 1962, when Rasch became professor, sociology was a relatively new branch of study.[60] A chair in sociology had been established in 1948, but it was not until 1958 that a Master's degree in sociology was introduced.

Kaare Svalastoga, who had held the chair in sociology since 1955, was in favour of examining questions raised in sociology by analysing questionnaires. He was furthermore of the positivistic school of thinking, and his ambition was to lay down rules on social conditions just as rules are laid down in natural sciences (Wolf, 2001, p. 297-298). Rasch was also of the positivistic school, and the Models for Measurement

[58] Acronym for: Annual Danish Aggregated Model. ADAM is the macroeconomic model of Denmark, which Ellen Andersen developed in her Doctoral thesis of 1975 (Ellen Andersen, 1975) (A Doctoral thesis is much more demanding than a PhD thesis). ADAM was immediately taken over by Statistics Denmark, and has since then been the official model of the central administration.

[59] In Danish in 1965: Stipendiat.

[60] Section 11 is based on Wolf (2001), Int. Christiansen (12.02.2001), Int. Groth (19.04.2002), Int. Niels-Erik Jensen (27.06.2001), Int. Karpatschof (23.04.2002), Int. Stene (08.10.2002).

were perfectly suited to analysing questionnaires. Moreover, the Models for Measurement were tools by which it is possible to detect whether data yields measurement of the kind met in natural sciences (Rasch, 1968'). Rasch's curriculum was therefore an almost perfect match to Svalastoga's point of views. Furthermore, as opposed to the students of economics, the students of sociology thought of Rasch's course in statistics as something useful for them. Groth says (Int. Groth, 19.04.2002):[61]

> *And since surveys are important in sociology, and since the Models for Measurement are designed to analyse them, the course was popular. At least in our year of sociology students, and, I dare say, also for the successive one. We were the students who started in 1963.*[62] *We were about 50 students.*

Groth also tells that some of the students of sociology really liked Rasch's lectures. He says (Int. Groth, 19.04.2002):[63]

> *We were quite a few who thought it amusing. Among other things because, being a student of sociology, it was interesting to know about more tangible matters - where something was true, and something was false. And where you could work out problems. It was also good that we were confronted with examples from psychology; tests in reading and the like. And also some from sociology.* [...] *What also amused some of us was that he was so committed. And at the same time so critical about the normal distribution.*

The students of sociology and the students of economics evidently had opposite points of view on the curriculum in statistics. Whereas the economists thought the Models for Measurement were immaterial, the sociologists thought them important. And whereas the economists thought the normal distribution and methods based on this distribution to be lacking in Rasch's curriculum, the sociologists thought Rasch's critical attitude was an amusing whim.

Benny Karpatschof who, in the mid 1960s was a teaching assistant in the Statistical Institute[64] and later an external lecturer in the Institute of Sociology, emphasizes that not only did the students of sociology like statistics, it was actually the favourite course of some of them. As Karpatschof sees it, Rasch was a success

[61] The text has been translated from Danish. The original text is given in Appendix E.

[62] At that time, statistics was a third year course, implying that Groth had tutorials in statistics in the academic year, 1965 to 1966.

[63] The text has been translated from Danish. The original text is given in Appendix E.

[64] In Danish: Undervisningassistent.

as a professor in the relationship to the sociologists. He made progress within the field of statistics as a professor of statistics should, and his developments were useful to the sociologists. Models for Measurement and specific objectivity have already been mentioned, but Rasch also developed a class of statistical models for analysing categorical data, namely 'Fordelingsanalyse ∼ Distribution analysis', and this class of statistical models was especially efficient when analysing questions raised in sociology (Int. Karpatschof, 23.04.2002).

Despite his good relationship with the sociologists, as Rasch approached retirement, they also became discontented with him. In 1968, a student protest dominated the University of Copenhagen. The students wanted influence on the curriculum and on the ways of academia. At the Institute of Sociology, an implication was that the balance of power was disturbed. Whereas the power had been centred around Svalastoga, it was now centred around the young teachers. And the young teachers were not in favour of positivism, so neither were the students. And since statistics was a positivistic minded course, it was no longer popular. Of course, it did not help that Rasch's course in statistics was difficult. Groth says (Int. Groth, 19.04.2002):[65]

> *But then came 1968, and the younger sociology students began to think that statistics was too heavy to deal with, and by no means necessary. And, it was considered to be a difficult course amongst the sociologist. There is no doubt about that. And, there were those who wanted to get rid of it. So, the development was that at some point in time the sociologists got their own course in statistics.*

After some 3 or 4 years, that is, in the early 1970s, the sociologists' course in statistics became quite watered-down.

[65]The text has been translated from Danish. The original text is given in Appendix E.

12 Rasch's scheme for the Statistical Institute

At the organizational level Rasch had a plan to turn the Statistical Institute into an interdisciplinary Institute with several departments, each department representing a branch of science.[66] Stene explains that this plan was a continuation of Rasch's involvement in interdisciplinary research (Int. Stene, 08.10.2002). As a statistical consultant, Rasch had cooperated with scientists from various fields he had hereby seen that such cooperation gave leave for new ways of thinking, both for Rasch and for the scientists he worked with; a fact that is apparent in the some 50 empirical papers Rasch published (see e.g. Rasch (1980)). It is therefore not surprising that Rasch tried to teach his students that statistical analyses ought to be carried out in cooperation with a scientist from the appropriate branch of science (c.f. Page 148), and neither is it surprising that Rasch tried to create an institute for interdisciplinary research.

However, Rasch also acted as a missionary for the Models for Measurement. The effect of this was that even though Rasch had the best intentions by planning for interdisciplinary research, to some extent it appeared as though his main purpose was to propagate the Models for Measurement. Mossin says (with a twinkle in his eye) (Int. Mossin, 24.09.2002):[67]

> *And I also think that he had a plan for his institute; that now there was to be this big breakthrough in sciences. The Models for Measurement was to propagate to all fields of science, and therefore, various scientific assistants were to be employed. Hence, Matthiessen's mission was demography, and I was employed as the economist, who was to transform economic theory entirely to Rasch's principles,* [...]*, it was this idea that all fields of science were to be pervaded by the Models for Measurement, and therefore, he had to have a versatile staff, including scientists from various disciplines.*

We shall not go into further details about Rasch's motive for attempting an interdisciplinary institute. The important thing is that it was a project of his, and that he pursued it at faculty meetings of the Faculty of Social Sciences. But, Rasch did not always attend the departments' meetings. In fact, he developed a tendency to shirk them, which might have been a contributory factor in the failure his ambition for an interdisciplinary institute. Erling Andersen evaluates Rasch's plan in an obituary

[66] Section 12 is based on Davidsen (2001), Wolf (2001), (Int. Bentzon. 27.08.1998), (Int. Christiansen, 12.02.2001), (Int. Niels-Erik Jensen, 27.06.2001), (Int. Mossin, 24.09.2002), (Int. Stene, 08.10.2002).

[67] The text has been translated from Danish. The original text is given in Appendix E.

(Andersen, 1980, p. 539-540):[68]

> *Also as a leader he worked large scale. He did not care much about practical details, but willingly pursued a comprehensive plan, often irrespective of the reality of the plan. It is characteristic that his plan for extending the Statistical Institute in the 60s included 4 independent departments and a proportionally large number of associates. Even in that period, which is now referred to as 'the happy sixties', this plan was unrealistic and probably hampered more specific co-operation with economists and sociologists. He had most success in establishing a demographic department, which re-established the close connection to the population statistics that was present at the beginning of the century.*

13 Projects and working climate at the Statistical Institute

Rasch may have failed in his ambition to create an interdisciplinary institute, but, he succeeded in initiating the project.[69] It seems to have been characteristic of Rasch that he was quite good at initiating things. At the Statistical Institute, Rasch managed to get several projects off the ground. He would get a group of people working on a problem, and, more importantly, he would keep an eye on them, approaching them to hear about their progress, and to give advice.

One of Rasch's principles was always to have young people working as assistants to help with the calculations, and he approached them for progress updates; sometimes a little too often in opinions. Lis Olsen, who made punch cards at the Statistical Institute, remembers that some of the young assistants, among these Allerup, at some point in time got a little tired of Rasch frequently dashing into their office, to inquire about their progress and efforts. So, they got hold of some bookshelves, and placed them such that young slim assistants such as themselves easily could get to their desks, whereas a stout senior professor such as Rasch would have to make an effort. This limited the overwhelming attention they were receiving (Olsen, 25.11.2002).

Rasch was also good at raising funds. Stene says (Int. Stene, 08.10.2002):[70]

> *It was quite typical of Rasch that he formed groups that worked with very*

[68] The text has been translated from Danish. The original text is given in Appendix E.

[69] Section 13 is based on Int. Christiansen (12.02.2001), Int. Niels-Erik Jensen (27.06.2001), Int. Mossin (24.09.2002), Int. Stene (08.10.2002).

[70] The text has been translated from Danish. The original text is given in Appendix E.

> *diverse projects.* [...] *And it was also this way that he raised money. When you can document that you have a scientific project and that you have a group of people working on it, then the money starts rolling in.*

According to Stene, Rasch was very good at writing applications. He wrote them out of the principle that you had to have a good project; that was the basis of every application. And then, it would not do to apply for some smallish amount. You should make a grand project and apply for a correspondingly grand amount of money (Int. Stene, 08.10.2002).

Though Rasch was quite good at initiating projects, he was, generally speaking, not that successful in finishing them. As to the application side of this, Stene says (Int. Stene, 08.10.2002):[71]

> *He was absolutely fabulous at expressing the fact that nothing definite had come up. He phrased this such that he got the application renewed. He was a brilliant manipulator on that score.*

When planning a carrier in academia, one needs to do research. On this account, Rasch was known to be helpful to his associates. For instance, Mossin tells that while he was an assistant professor at the Statistical Institute, Rasch got him going on his first publication (Mossin, 1972). Rasch had a lot of connections around the world, and he had contacted the manager of a firm doing research in consumers' behaviour, solely with the view to getting Mossin data. A meeting between the manager, Rasch and Mossin was arranged, and the terms upon which Mossin was to use these data were agreed upon. Furthermore, when Mossin's project turned out to be computationally demanding, Rasch arranged it such that Mossin got computational assistance from Bjarne Andersson, who was very good at programming the computers of that time. Mossin, who after a couples of years at the Statistical Institute, got a position at the Institute of Economics, says about Rasch (Int. Mossin, 24.09.2002):[72]

> *First of all, I think he was a man of strong character, and he was one of these old-fashioned professors, who no longer exists. He had a very domineering personality. But, he was very generous.* [...] *Rasch had this caring manner towards his staff.* [...] *When I was employed at the Institute of Economics, there was very few assistant professors,*[73] *and few PhD students.*[74]

[71] The text has been translated from Danish. The original text is given in Appendix E.
[72] The text has been translated from Danish. The original text is given in Appendix E.
[73] In Danish: Videnskabelig assistent.
[74] In Danish: Kandidatstipendiat

> *There were Niels Thygesen and Erling Olsen, who were employed as assistant professors. And the intention was that they were to write their doctoral theses. But the idea of talking to people and helping them with their project was totally foreign to these old professors, because thesis writing was an assignment they had.* [...]
> *But, in that area, Rasch had more warmth in his manner. And the domineering personality, which some complained about, but, he had stature as a human being, and generosity and things like that. So, therefore, I am a great fan of Rasch.*

For those projects that did result in a paper, Rasch showed his helpful side once more by reading them through. But, his domineering personality showed itself alongside, and colleagues of Rasch needed to be a little thick-skinned. Stene says (Int. Stene, 08.10.2002):[75]

> *If you had a paper in the writing, Rasch gave the most fantastic comments.* [...] *He was indeed thorough, and wrote a lot of comments in the margin. And then, when he came back with it, then Rasch summoned you. He could be fierce. But, it was constructive. I did not feel angry or embittered. I felt enriched. But, he was so very rude.*

All in all, the working climate at the Statistical Institute, was considered to be quite good. Furthermore, with all these people working on various problems, talking to one another, and Rasch mingling with them all, the Statistical Institute was a fruitful environment for sciences. Stene says (Int. Stene, 29.10.2002):[76]

> *I interpreted the environment at Sankt Peders Stræde* [*The location of the Statistical Institute*] *as an active and dynamic environment, where people discussed various scientific problems. And, during the first years, when Rasch was there regularly, you had this feeling that it was a scientific unit.*

14 Consulting and further developing the Models for Measurement

By time,[77] some of the everyday task became tedious to Rasch. He had little patience with administrative duties such as attending faculty meetings, and he therefore developed a tendency to shirk them by sending his associates. To the question whether

[75] The text has been translated from Danish. The original text is given in Appendix E.
[76] The text has been translated from Danish. The original text is given in Appendix E.
[77] Section 14 is based on Int. Niels-Erik Jensen (27.06.2001), Int. Karpatschof (23.04.2002), Int. Stene (08.10.2002).

Rasch got off lightly from his duties as a professor, Niels-Erik Jensen, who today is an associate professor, and who, as an associate at the Statistical Institute, has been giving lectures in statistics since the mid 1960s, says (Int. Niels-Erik Jensen, 27.06.2001):[78]

> *I wouldn't say that. But, you could say that he got off lightly from his administrative duties. And, he also got off relatively lightly from his teaching, because he had money to hire tutors*[79] *to lecture in his place, for instance, my first lecture was as a tutor.*[...] *Rasch was interested in sciences, but not interested in how the study was organized - not at all. It was my impression that he would rather avoid that.*

Thus, Rasch was neither overburdened with administrative duties, nor with teaching. This must, however, not be construed to mean that Rasch was lazy and did nothing with his time. It was just that his scientific interests were centred away from faculty matters. Christiansen says (Int. Christiansen, 12.02.2001):[80]

> *So his scientific studies and his external relations surfaced at international seminars and at his study circle for educationalists. Quite a few Americans visited. Americans who visited us to work with test-related problems within psychology.*

In Appendix C, a list of visitors to the Statistical Institute in the period 1964 to 1972 is given. A substantial number of these people worked with the statistical theory that Rasch had developed, namely the Models for Measurement and specific objectivity. It is therefore safe to say that Rasch used his professorship to further the sciences he had developed. And it is likewise safe to say that he enjoyed doing so.

Before Rasch was appointed professor, his main occupation was to be a statistical consultant. This work did not stop just because he became a professor. Karpatschof says (Int. Karpatschof, 23.04.2002):[81]

> *He was an eccentric character in all manners. And, for better or for worse he pursued the things he was enthusiastic about. And, I suppose he sincerely tried to develop something that was useful for both the economists and the sociologists. But, then he reached a certain point.* [...]
> *He found out that he was not able to contribute with anything important,*

[78] The text has been translated from Danish. The original text is given in Appendix E.
[79] In Danish: Undervisningsassistenter
[80] The text has been translated from Danish, see e.g. Appendix E.
[81] The text has been translated from Danish. The original text is given in Appendix E.

and then he returned to his main purpose in life, namely to be an consultant. He was a magnificent consultant.

In the early 1960s Rasch consulted at the Danish Institute for Educational Research on Fridays, at the State Serum Institute on Monday afternoons and at the Military Psychology Service on Thursday afternoons. In time, as Rasch got older, he restricted his consulting to that at the Danish Institute for Educational Research (Int. Stene. 08.10.2002).

In addition to the places mentioned where Rasch consulted on a regular basis, various scientists, ministries and firms also consulted him. For instance, in the late 1960s a large-scale investigation into the effect of speed limits on the number of traffic accidents was carried out, and the Ministry of Traffic consulted Rasch. Rasch employed a young statistician, educated at the University of Copenhagen, Faculty of Mathematics and Natural Sciences, Nils Kousgaard, to do the necessary computations. Afterwards, Kousgaard was employed as an assistant professor,[82] later associate professor. After Rasch's retirement, Kousgaard was the string-puller in establishing a working relationship between statisticians and economists. He introduced regression analysis as a course in statistics, and, in general, he was good at getting people to talk to one another, instead of trotting out their favourite ideas (Int. Olsen, 25.11.2002).

There is no doubt that Rasch enjoyed consulting, and it appears that it was here that he discovered new theory. For instance, the the Models for Measurements were developed through his consulting work (see Andersen and Olsen (2001)). Likewise was his development of a model for analysis of growth (see Chapter 2 of this thesis). And, in connection with the analysis of traffic accidents, Rasch developed specific testing (see the Introduction and Chapter 5 of this thesis). It was as if the problems he faced when analysing data forced him to develop new theories; and he liked that. However such theoretical developments remained in an appendix or in hand written notes, as Rasch went on to another project. Rasch was said to be a real scientist who took joy in developing new things, and to marvel at theory. But, he did not enjoy finishing a paper for publication.

[82] In Danish: Videnskabelig assistent

15 Approaching retirement

All in all, Rasch had a lot of projects going.[83] But one of the projects that he had some difficulty getting off the ground was rewriting his first textbook, Rasch (1966). Stene says (Int. Stene, 08.10.2002):[84]

> *Rasch often talked about making some new textbooks. But, he never got around to it. Therefore we decided, a group consisting of Rasch, Ulf Christiansen, Matthiessen and I, that we would rewrite that textbook.*

In the beginning Rasch was quite enthusiastic about this plan, and he eagerly participated in the discussion about the contents of the first chapters. In time, Niels-Erik Jensen got involved in the project, and Matthiessen withdrew, and in a sense, Rasch withdrew too. Stene says (Int. Stene, 08.10.2002):[85]

> *The first chapters were discussed at length, but later Rasch lost interest in the project, and went away. He travelled to Chicago.*

There is no apparent explanation for why Rasch lost interest in the project. It might be that Rasch saw little point in continuing developing the course in statistics after the sociologists stopped attending it, but, other explanations are possible. For instance, that Rasch found it much more interesting to be in Chicago and talk about his favourite topic, namely the Models for Measurement, or, it might be due to Rasch's ill-health in the late 1960s. Whatever the reasons, the effect was that the first chapters of the second textbook were discussed at length, but the remaining half of the book was discussed in the course of just one afternoon. The new textbook, in two volumes, was published in 1968 and 1969 respectively. Appendix A gives an outline of the contents of the second textbook.

Rasch's lack of interest in the new textbook seems to have been but one indicator of his decreasing interest in the Statistical Institute. He was certainly ill, and on some occasions even hospitalised, but, he also had a rather large consumption of alcohol. Whatever the reason, his decreasing interest meant that he spent less and less time at the Statistical Institute in favour of his summer residence on Læsø. Stene says (Int. Stene, 08.10.2002):[86]

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[83] Section 15 is based on Int. Niels-Erik Jensen (27.06.2001), Int. Karpatschof (23.04.2002), Int. Mossin (24.09.2002), Int. Stene (08.10.2002).

[84] The text has been translated from Danish. The original text is given in Appendix E.

[85] The text has been translated from Danish. The original text is given in Appendix E.

[86] The text has been translated from Danish. The original text is given in Appendix E.

> *As it happened, after 1968 he did not turn up regularly at the institute.* [...] *He was at Læsø. But then, about every third month he turned up and stayed for a couple of days. But otherwise he was at Læsø, and then we could phone him there.* [...] *He had maladies. He drank like a fish. And then, as it happened, he fell down the stairs on the ferry sailing to Læsø. He had maladies that he used as excuse for not turning up at the institute.*

Actually, when Rasch fell down the stairs on the ferry, he broke his neck but, as this was not discovered until later the neck healed incorrectly and this gave him a stooping figure (Int. Allerup, 26.02.1998).

16 Retirement

In 1971 Rasch retired as professor.[87] His former student, Erling B. Andersen was appointed as his successor to the professorship. Unfortunately for Rasch, he did not have much income to live on after his retirement. He had been professor for too short a period to receive a pension, and while he worked freelance as a statistical consultant, he had not accrued any savings; though his income at times was very good indeed. In the end, something was worked out, and Rasch received a small pension corresponding to the 10 years he had been professor. Fortunately for Rasch, the research councils were generous to him. Arne Jensen says (Int. Arne Jensen, 08.02.2001):[88]

> *And the research councils were very generous to him. At the National Institute of Social Research, Henning Friis, was very helpful. And that was actually a funny thing, because if there was somebody Rasch could not stand, it was Henning Friis. But he thought* [Friis] *that it was proper to support that man* [Rasch].

Rasch's retirement did not stop him from doing research. He continued to work within his fields of interest, namely the Models for Measurement and specific objectivity. For instance, in 1972 Rasch visited Ben Wright in Chicago and lectured on the Models for Measurement. It was on this occasion that Rasch met Andrich (c.f. the Andrich-Interview).

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[87] Section 16 is based on Int. Allerup (26.02.1998), Int. Arne Jensen (08.02.2001), Int. Niels-Erik Jensen (27.06.2001), Int. Stene (08.10.2002).

[88] The text has been translated from Danish. The original text is given in Appendix E.

Actually, Andrich is an example of how, even in his retirement, Rasch was able to inspire young scientists, because Andrich still considers Rasch's ideas to be genius, and Andrich has made his own contributions within this field. When Andrich met Rasch in 1972, Andrich was a graduate student, and Rasch was newly retired. Wright, being Chair of Andrich's PhD committee, encouraged Andrich to visit Rasch in Denmark on his way back to Australia in 1973. This marked the beginning of a friendship that lasted for the rest of Rasch's life. In 1974, Andrich arranged for Rasch to be visiting professor at the Departments of Mathematics and Education at the University of Western Australia, where Andrich held a position. The stay lasted seven months. On his way back to Denmark, Rasch visited India where he had scientific connections (c.f. the Chapter, *Georg Rasch's Growth Model*). Hence, Rasch was also a globetrotter in his retirement.

Andrich writes about what made Rasch such an inspiration; especially regarding the Models for Measurements (Andrich, 2002):

> *All this exposure to Rasch made me sense that here was an extraordinary character and that his character played an important role, not just in his discovery and insight,* [...] *Rasch displayed a character that seemed to me to be absolutely consistent, impeccably rigorous, totally uncompromising, and most importantly, passionate. I saw him cry on more than one occasion when he explained his hopes and visions for his work because he believed his insight went beyond the matter of social science measurement, he believed the ingredient of an invariant comparison was an integral part of the possibility of knowledge itself.*

Andrich is not a one-off example of young scientists who were inspired by Rasch.

In his retirement Rasch continued to work on the project on traffic accidents; actually he worked on this project until 1979 (Andersen, 1980). He would, from time to time, visit the Statistical Institute. On such occasions he would sit in the library, because no office was allocated to him; a fact which disappointed him greatly (Int. Stene, 08.10.2002).

Ben Wright, who was also inspired by Rasch, and was at the University of Chicago, arranged for Rasch's book of 1960 about the Rasch models to be reprinted by the University of Chicago Press. A copy arrived for Rasch just after he died on 19th of October, 1980.

17 Summa summarum

Rasch was originally trained in mathematics, and his hopes were to make a future in academia within this field of science. As this future was denied him, he turned to statistics. A side effect was, however, that Rasch came to be of the opinion that influential mathematicians thwarted his opportunities of ever holding a chair in mathematics at the Faculty of Mathematics and Natural Science at the University of Copenhagen. He was further confirmed in this opinion when he found out that he was not even permitted to apply for the chair in statistics that was established in 1958 in the Faculty of Mathematics and Natural Sciences. When he later applied for the chair in statistics at the Faculty of Social Sciences, a chair which he had not shown interest in before, he was in an embittered state of mind.

When Rasch took up his professorship in 1962, he had developed a few favourite ideas, namely that the normal distribution and methods based on this distribution were careless and used unscientifically and should be avoided. He was also of the opinion that every dataset was to have it's own tailored statistical model and that everybody ought to use the Models for Measurement whenever possible. Rasch decided to change the existing statistics curriculum in the Faculty of Social Science, and the corner stone of his own new curriculum corresponded to his favourite ideas. He taught statistics to students of sociology and students of economics. However, while Rasch's new curriculum was almost perfect for the students of sociology, it was thought by many that it could hardly be more inappropriate for the students of economics.

As to Rasch's activities at the Statistical Institute, he was said to be very domineering, but at the same time caring towards his associates, helping them to get their projects going. He was a professor of the old school, but well liked by his colleagues. He was good at initiating projects and creating a productive working climates, where the associates and the young scientific assistants talked to one another, discussing sciences, statistics and the Models for Measurement.

18 Appendix A. Outline of the contents of Rasch's textbooks

In the early and mid 1960s,[89] Rasch's curriculum for the basic course in statistics was contained in his first textbook, namely Rasch (1966), and combined with a booklet on the normal distribution written by Jørgensen (Jørgensen, 1957). Later, Rasch's curriculum was contained in his second textbook, in two volumes, published in 1968 and 1969, respectively (Rasch, 1968), (Rasch, 1969). Furthermore, as supplementary reading Rasch suggested a book that gave an experimental introduction to the theory of probability written by Rasch's friend, J.E. Kerrich (Kerrich, 1950). Actually, none of these books were written by Rasch, not even the books that bear his name. The first textbook was essentially an account that Christiansen gave of Rasch's lectures. And the two volumes of the second textbook titled *First volume of G. Rasch's textbook on theoretical statistics* and *Second volume of G. Rasch's textbook on theoretical statistics*, respectively, were, with some exceptions, written by Christiansen and Stene. There is, however, absolutely no doubt that the contents of these textbooks represent Rasch's fundamental view on what was appropriate for a basic course in statistics (Int. Christiansen, 12.02.2001), (Int. Kærgård, 26.02.2001). An outline of the contents of Rasch (1966) is given in Table 4.1 while an outline of Rasch (1968) and Rasch (1969) is given in Table 4.2.

Both Rasch's first and second textbooks in statistics start with rather detailed sections linking together experiments, such as toss of coins and the corresponding probabilistic models. In doing so, Rasch tried to communicate his general view of statistics, namely that data was generated by some mechanism or other, and that the goal was to uncover this mechanism (c.f. Section 7).

Both of Rasch's textbooks on statistics include the Models for Measurement and specific objectivity (c.f. Section 6). In the first textbook, this is covered in the quite extensive chapter titled, *Models for Measurement* (c.f. Table 4.1), and in the second textbook it is included in chapters 11 through 13 (c.f. Table 4.2). Except for the multiplicative Poisson, the statistical models of these chapters are mostly used to analyse questionnaires and the like, and it is safe to say that these models rarely, if ever, are included in basic textbooks in statistics. Also the chapter *Distribution Analysis* can be found in Rasch's textbooks only. Distribution Analysis can be viewed as a sort of appendix to the Models for Measurement, where populations instead

[89] Appendix A is based on Rasch (1966), Rasch (1968), Rasch (1969), Int. Christiansen (12.02.2001), Int. Stene (08.10.2002)

TABLE 4.1. Contents of the first textbook, Rasch (1966).

Chapter		pages
Introduction Lectures		15
General Theory, I	The concept of chance	114
	Axioms of probability	
	The law of the binomial	
	Mean, variance, generating functions	
	The Poisson Law	
	The law of the multinomial	
Models for Measurement	The dichotomous Rasch model	109
	The multiplicative Poisson model	
	The polytomous Rasch model	
	Axiomatic for Models for Measurement	
General Theory, II	Continuous distributions	55
	Distribution function	
	Mean, variance	
	Transformation of distributions	
	Control of continuous models	
Distributions analysis		24
Time series analysis		15
Stochastic processes		18

of individuals are analysed. In effect, comments made to the chapter on Models for Measurement pertain, on some level, to the chapter on Distribution Analysis as well. But only on some level, since Distribution Analysis can be seen in line with analysis of categorical variables using log linear models.

As it was, regression analysis did not enter into Rasch's first textbook (Rasch, 1966). In the second textbook, chapter 9 deals with regression analysis and analysis of variance. But even though the part about regression analysis covers 40 pages, the coverage of this topic was judged to be modest (Weibull, 1970). Supposedly, this judgment is made relative to the total number of pages, namely 732.

The normal distribution, on the other hand, is mentioned in both the first and the second textbooks. In the first textbook the normal distribution is only mentioned on some three occasions, but when the first textbook was used, the normal distribution was covered in the booklet, Jørgensen (1957). In the second textbook, the normal distribution is well covered.

In a review of Rasch's second textbook, it was pointed out that the mathematical level of Rasch's textbook was too high (Weibull, 1970). It seems the students in

TABLE 4.2. Contents of the second textbook, Rasch (1968) and Rasch (1969)

		Chapter	pages
Vol I	1.	The concept of chance and data	30
	2.	Axiomatic and distributions	98
	3.	Mean, Variance and Generating functions	30
	4.	The law of the binomial	58
	5.	The Poisson law and the Poisson process	50
	6.		30
	7.	Descriptions of data by the normal distribution	70
	8.	Analysis of data which may not be described by the normal distribution	31
Vol. II	9.	Analysis of variance and regression analysis	92
	10.	Sampling techniques	50
	11.	The multiplicative Poisson model	38
	12.	The dichotomous Rasch model	48
	13.	General properties of the models of chapter 11 and 12	18
	14.	Distribution analysis	38
	15.	Time series analysis	38

general agreed with this, but that Rasch was quite content with the fact that his course in statistics was difficult (Int. Christiansen,12.02.2001). The review also drew attention to the fact that Neyman Pearson test theory was not even mentioned, and that the book might as well have been written in the 1930's, since no new statistical theory except that of Rasch's was included. Since Rasch was known to dislike the Neyman Pearson test theory (Int. Allerup, 26.02.1998) this may be seen as yet another example of Rasch's strong opinions on statistical theory, and that he had few hesitations about letting these opinions dictate his curriculum.

19 Appendix B. List of personnel

Tables 4.3 trough 4.4 displays the permanent positions available at The Statistical Institute,[90] and the surnames of those who Rasch employed in these positions. As most positions are specific to Danish Universities of the 1960s the list is given in Danish. An introduction to the various positions therefore preludes the list.

It must furthermore be mentioned that the permanent positions in Table 4.3 do

[90] Appendix B is based on Årbog (1958-63), Årbog (1963-64), Årbog (1964-65), Årbog (1965-66), Årbog (1966-67), Årbog (1967-68), Årbog (1968-69), Årbog (1969-70), Årbog (1970-71), Årbog (1971-72).

TABLE 4.3. List of positions at the Statistical Institute

1963-64	1 bestyrer	Rasch
	1 amanuensis	Matthiessen
	1.5 laboratory assistent	Bak-Andersen, Fischer
1964-65	1 bestyrer	Rasch
	1 amanuensis	Matthiessen
	2 videnskabelige assistenter	Christiansen, Mossin
	1.5 laboratorie assistenter	Bak-Andersen, Fischer
1965-66	1 bestyrer	Rasch
	1 amanuensis	Matthiessen
	2 videnskabelige assistenter	Mossin, Stene
	2.5 laboratorie assistenter	Bak-Andersen, Olsen
1966-67	1 bestyrer	Rasch
	1 afdelingsleder	Matthiessen
	3 videnskabelige assistenter	Dollerup/ Lundtorp, Mossin, Stene
	2.5 laboratorie assistenter	Bak-Andersen, Hallgren, Olsen
1967-68	1 bestyrer	Rasch
	1 afdelingsleder	Matthiessen
	3 videnskabelige assistenter	Jensen, Mossin, Stene
	2 laboratorie assistenter	Bak-Andersen, Hallgren, (Olsen)
1968-69	1 bestyrer	Rasch
	1 afdelingsleder	Matthiessen
	4 videnskabelige assistenter	Hansen, Jensen, Kousgaard, Stene
	2 laboratorie assistenter	Bak-Andersen, Hallgren, (Olsen)

not include everybody who worked at the Statistical Institute. Rasch managed to get other funds for the institute, and this capital was especially used to employ young students with a view to having computational assistance. These 'positions' and those who were employed in them are not mentioned in the yearbooks of the University of Copenhagen, and it has therefore not been possible to reconstruct this information.

Translation of the Danish names for the permanent positions:

Adjunkt: Assistant professor.
Afdelingsleder: Associate professor with administrative responsibility.
Amanuensis: Assistant professor or associate professor.
Bestyrer: The head of the Statistical Institute. The 'Bestyrer' of the Statistical Institute was a professor.

TABLE 4.4. List of positions at the Statistical Institute

1969-70	1 bestyrer	Rasch
	1 afdelingsleder	Matthiessen
	4 vid.ass.	Hansen, Jensen, Kousgaard, Stene
	1 kandidat stipendiat	Allerup
	3.5 lab ass	Bak-Andersen, Bakke, Olsen
		Hallgren/Sørensen
1970-71	2 professorer	Matthiessen, Rasch
	1 docent	Stene
	1 afdelingsleder	Jensen
	3 videnskabelige assistenter	Andersen, Hansen, Kousgaard
	3.5 lab.ass.	Bak-Andersen/Blakvid, Bakke,
		Olsen, Sørensen
1971-72	2 professorer	Matthiessen, Rasch
	1 docent	Stene
	3 lektorer	Hansen, Jensen, Kousgaard
	1 adjunkt	Andersen
	2 kand. stip.	Allerup, Toft-Nielsen
	1 fondslønnet	Bez
	3.5 tap	Bakke, Blakvid, Olsen, Sørensen

Docent: Associate professor. Higher rank than an amanuensis.
Fondslønnet: A foundation gave money to the Statistical Institute. The money was administered by the University of Copenhagen.
Kandidat stipendiat: Ph.d. student.
Laboratory assistent: Assistant
Lektor: Senior lecturer. Higher rank than an amanuensis, but lower rank than a 'Docent'.
Professor: Professor.
Tap: Non-academic staff.
Videnskabelig assistent: Research assistant or assistant professor.

20 Appendix C. List of guests at the Statistical Institute

1963-64 F. Tschudi from University of Oslo.[91]

B. Wright, Department of Education, University of Chicago.

Both have worked under Rasch's guidance.

[91] Appendix C builds on Årbog (1958-63), Årbog (1963-64), Årbog (1964-65), Årbog (1965-66), Årbog (1966-67), Årbog (1967-68), Årbog (1968-69), Årbog (1969-70), Årbog (1970-71), Årbog (1971-72).

1964-65 B. Wright, Department of Education, University of Chicago. Wright has worked under the study guidance of Rasch.

1965-66 Monroe C. Sirken from Chief Division of Health Records Statistics, National Center for Health Statistics, U.S.A.

I.M. Moryama, PhD, Chief Office of Health Statistics Analysis, National Center for Health Statistics, U.S.A.

Lincoln H. Day, Yale University.

1966-67 Bruce Choppin, London.

H. Wold, Stockholm.

Abdel M. Shafei, North African Demographic Center.

Ahmed E. Sarhan, The Institute of Statistical Studies and Research, University of Cairo.

Gerhard Fischer, Vienna.

Ben Wright, Chicago.

(Wright and Fischer have both worked under the guidance of Rasch.)

1968-69 Tarow Indow, Kyoto University, Japan.

S. Mednick, New School, New York.

R. J. Kearns, Berkeley University, USA (working under Rasch's guidance.)

K. Bez, Ganhati University, Indien. Study of the Models for Measurement.

R.A. Henin, University of Khartoum, Sudan.

1969-70 Gerhard Fischer, Institute of Psychology, University of Vienna.

K. Bez, Ganhati University, India (study of the Models for Measurement under Rasch's guidance.)

Etienne van de Walle, Office of Population Reasearch, Princeton University.

1970-71 Trygve Haavelmoe, University of Oslo.

John Hajnal, London School of Economics.

T.W. Anderson, Stanford University, U.S.A.

Ralph Ginsburg, University of Pennsylvania.

Ravenholt, Director of Technical Assistance Bureau, Agency of International Development, State Department, Washington D.C, U.S.A.

21 Appendix D. Interviews

Appendix A contains background information about the people who so kindly told the present author about their memories of Rasch. Special emphasis is on their connection to Rasch.

Allerup, Peter. Interview 26.02.1998. Peter Allerup is a statistician by training. He became associated with Rasch in the 1960s, and he worked with him at both the

Statistical Institute and the Danish Institute of Educational Research, where Rasch was affiliated as a consultant. Allerup has continued to work with the Rasch Models at the Danish University of Educational Research, where he now holds a position as professor of statistics.

Ellen Andersen E-mail 29.08.2002. Ellen Andersen is an economist by training. She was employed at the Institute of Economics in 1965, and in 1973 Ellen Andersen was appointed professor of empirical economics. Rasch was a professor at the Statistical Institute from 1962 to 1971, and the intention was that statistics was to be an auxiliary subject for economics and sociology. Hence, Ellen Andersen came to know Rasch through their work and cooperation together at the same faculty. (see. e.g. Chapter 4 of this thesis).

Andrich, David. Interview. 04.02.2002 and 06.02.2002, Perth, Western Australia. Andrich met Rasch in 1972 in Chicago. At this point in time, Andrich was a graduate student, and Rasch was newly retired. Rasch was visiting Ben Wright in Chicago, and Wright, who was the Chair of Andrich's PhD committee, persuaded Andrich that, he on his way back to Australia in 1973, he should visit Rasch in Denmark. Andrich stayed with Rasch for a week or so, and in 1974 Andrich arranged for Rasch to be a visiting professor for seven months in the Departments of Mathematics and Education at the University of Western Australia, where Andrich held a position. Andrich visited Rasch again in 1975, and in 1977 Andrich spent five months at the Danish Institute of Educational Research, where Rasch was still affiliated as a consultant. In 1979, Andrich visited Rasch with the purpose of interviewing him. The interview Andrich conducted on this occasion is the one referred to throughout this article as the Andrich-Interview. Andrich is now professor of Education at Murdoch University in Western Australia.

Bentzon, Michael Weis. Interview 27.08.1998. Bentzon got to know Rasch in the late 1940s, when Bentzon attended the course Rasch gave on statistics at the Faculty of Mathematics and Natural Sciences. Bentzon thereafter came to work for Rasch at the State Serum Institute, where Rasch worked as a statistical consultant. Their working relationship continued throughout the period Rasch was with affiliated the State Serum Institute, that is, until the late 1960s. Bentzon and Rasch also saw each other socially.

Christiansen, Ulf. Interview 12.02.2001. Christiansen met Rasch shortly after Rasch

was appointed professor of statistics, and Christiansen was among the first staff members Rasch employed. After Christiansen got another job in the mid 1960s, he maintained a connection to the Statistical Institute. He was, for example, one of the authors of Rasch's two volume second textbook, namely Rasch (1968) and Rasch (1969).

Groth, Christian. Interview 19.04.2002. In the early to mid 1960s Groth was a student of sociology, and as such, he attended Rasch's lectures on statistics in the mid 1960s. He later became an instructor at the Statistical Institute, where Rasch was a professor of statistics. Today, Groth is an economist and he is an associate professor at the Institute of Economics.

Arne Jensen Interview 08.02.2001. Arne Jensen became associated with Rasch in the early 1940s when he began to work for Rasch at the State Serum Institute. In 1963, Arne Jensen was appointed professor of statistics at the Polytechnical College (today the Danish Technical University). Arne Jensen was a good friend of Rasch throughout Rasch's life, and they saw each other socially. Both Arne Jensen and Rasch had weekend cottages on the Danish island of Læsø.

Niels-Erik Jensen Interview 27.06.2001. Niels-Erik Jensen became affiliated with the Statistical Institute in the mid 1960s as an tutor. Later Rasch employed him to give lectures in a course on difference and differential equations. In 1968 Niels-Erik Jensen was appointed to one of the permanent positions available at the Statistical Institute. He soon became involved in the administrative duties, and became associate professor with administrative responsibilities[92] in 1971. Niels-Erik Jensen still (in 2002) holds a position as associate professor, though the Statistical Institute today comes under the Institute of Economics.

Karpatschof, Benny. Interview 23.04.2002 and 13.01.2003. As a student of psychology, Karpatschof attended the lectures Rasch gave on statistics for the psychologists (c.f. Chapter 1 of this thesis). Later Karpatschof was employed at the Military Psychology Service and the Danish Institute of Educational Research where Rasch consulted, and this way Karpatschof got to know Rasch. Afterwards, Karpatschof became affiliated the Statistical Institute as instructor. Karpatschof is today associate professor at the Institute of Psychology, University of Copenhagen.

[92] In Danish: afdelingsleder.

Kærgård, Niels. Interview 26.02.2001. As a student of economics, Kærgård attended Rasch's course on statistics. Later, Kærgård became affiliated with the Statistical Institute as an tutor. Today, Kærgård is professor of Agricultural Economics at the Royal Veterinary and Agricultural University.

Milhøj, Poul. Interview 04.07.2002. When Rasch was appointed professor, Milhøj ran the course, 'Danmarks statistik', at the Institute of Economics. The course was intended to be a thorough description of public matters. After Rasch was appointed professor, he approached Milhøj, presumably with a view to discussing possible applications of theoretical statistics in economy. But, after a while the discussion centred around sciences in general. In 1963, Milhøj was appointed professor at the Copenhagen Business School.

Mossin, Axel. Interview 24.09.2002. As a student of economics, Mossin attended the lectures Rasch gave as professor of statistics (c.f. Chapter 4 of this thesis). When Mossin graduated in 1965 Rasch employed him at the Statistical Institute, where Rasch held a position as a professor. Some years later, Mossin got a position at the Institute of Economics. Today, Mossin is an associate professor at the Institute of Economics.

Olsen, Lis. Interview 25.11.2002. Olsen started working at the Statistical Institute in the mid 1960s as a non-academic staff member. More specifically, she made punch cards. Olsen is still (year 2002) associated with the former Statistical Institute.

Stene, Jon. Interview 08.10.2002 and 29.10.2002. In 1963, Stene was employed at the Danish Institute of Educational Research. As a statistical consultant, Rasch worked at the Danish Institute of Educational Research every Friday, and it was here that Stene got to know Rasch. Stene became affiliated with the Statistical Institute in 1964 as a tutor, and in 1966, he got a position at the Statistical Institute. He soon became deeply involved in teaching and the writing of Rasch's second textbook. Furthermore, most of the unpublished papers 'written' by Rasch on the Models for Measurement have been thoroughly prepared by Stene. Today Stene is retired, but he held a position at the Statistical Institute until 2000 (In 1996 the Institute of Economics subjugated the Statistical Institute, implying that technically speaking, Stene held a position at the Institute of Economics from 1996 to 2000)

Ølgaard, Anders. Interview 15.11.2000. In 1962, Ølgaard was temporarily appointed

professor.[93] In 1966, this professorship was fully established, and in the same year, Ølgaard defended his doctoral thesis, a thesis that is judged to be the most substantial Danish contribution to economic growth theory of that time (Kærgård, 2001, p.167). Ølgaard has been one of the most prominent figures in the history of the Institute of Economics since its establishment in 1958 (Kærgård, 2001).

22 Appendix E. Quotes

The Original text to the quote on Page 136: Ja! Og det syntes Hald også dengang. Det jeg kan huske, er, at Hald prøvede på at få Rasch til at skrive noget. Han havde ikke skrevet noget. Hans indsats i forbindelse med konsulentopgaverne var jo ikke noget, der kunne bruges som argument for et professorat, og jeg tror, at Hald gjorde alt hvad han kunne for at få Rasch til at skrive noget. Men Rasch gjorde det altså ikke før i 60 med målingsmodellerne. Og det var det første egentlig væsentlige skrift, arbejde fra Rasch's side. Der var nogen artikler selvfølgelig.

The Original text to the quote on Page 137: Matematikerne ville jo have Hald og ikke Rasch, og vi var alle sammen enige om, at Rasch var den der havde sat det hele i gang, og at det var ham, der skulle være professor.

The Original text to the quote on Page 139: Lykke Jensen var jo økonom, og han interesserede sig for de statistiske metoder og så videre, der var gængse, og som vidt er blevet ved med at være standard inden for økonomisk forskning og økonometri, og det havde Rasch altså mildest talt ikke.

The Original text to the quote on Page 140: Et mindretal, P. Nørregaard Rasmussen, mener at Lykke Jensen har dokumenteret sin kompetence, idet mindretallet fremhæver, at det opslåede professorat i statistik henhører under det rets- og statsvidenskabelige fakultet. Undervisningen er for økonomistuderende og vejledning og samarbejde indenfor fakultetet bliver for og med økonomer. Mindretallet lægger under henvisning hertil særlig vægt på, at ikke alene er Lykke Jensen økonom af uddannelse, men har tillige i alle sine arbejder vist interesse for en række for økonomi vigtige statistiske metoder og i anvendelse af statistiske betragtningsmåder har han næsten undtagelsesfrit vendt sig mod økonomien.

The Original text to the quote on Page 140: Og det gjorde et stort indtryk på politterne. Ikke fordi vi overhovedet ikke drak spiritus, men vi kendte overhovedet

[93] In Danish: konstitueret.

ikke manden, og vi tænkte: det var alligevel fantastisk. Men der, jeg vil hen med det, er, at han var altså helt anderledes end os andre. Og det, som der så skete, det var jo altså, at han blev ansat, men man kan roligt sige, at han aldrig rigtig kom på bølgelængde med de fleste af os, og jeg vil måske sige, at det var han ikke engang interesseret i.

The Original text to the quote on Page 141: Og det var, skal vil ligesom sige, starten til en konflikt som bare blev ved. Han sagde, at de der økonomer ikke fattede hvad de lavede. Han mente, at de skulle anvende andre metoder end regressionsanalyse.

The Original text to the quote on Page 141: Det er også udtryk for Rasch's niveau; han krøb ikke for nogen.

The Original text to the quote on Page 142: Halds elever kunne varians- og regressionsanalyse på rygmarven, og der var mængder af økonomiske data i øvelsesopgaverne, således et tykt hæfte med data fra enkelte landbrugsbedrifter. I en hjemmeopgave estimerede jeg efterspørgelsesfunktion for diverse frugt og grønt baseret på gartneriernes salgsstatistik. [...] Jeg følte mig velforsynet med statistisk teori og økonometrisk metode.

The Original text to the quote on Page 143: Og du har sikkert hørt, at det foregik sådan ret kaotisk, fordi Rasch havde jo en ide om at han ville revolutionere statistikundervisningen, og der var ikke nogen lærebøger han kunne bruge [...] Planen var vist nok, at det skulle foregå fortløbende, men efterhånden gik der jo længere og længere tid mellem forelæsningerne og at det renskrevne notat forelå.

The Original text to the quote on Page 144: Der blev lavet delingen mellem lille og stor statistik, hvor Hald underviste dem, der havde stor statistik. [...] Og så var der lille statistik, og det var E. Lykke Jensen, der stod for det. Og det var det fag, de fleste gik til[...] Men så kom vi altså over til dette her med Rasch og det var så alle, der blev udsat for noget, som de i hvert tilfælde syntes var ikke sådan nemt. Der var selvfølgelig nogle, som syntes, det var udmærket, men det var altså et mindretal.

The Original text to the quote on Page 144: Statistik blev betragtet som utroligt svært; det med at bruge matematik i det omfang. Også økonomiundervisningen, den var jo meget meget mindre formaliseret. Og selv meget svag formatering blev af de studerende opfattet som noget der var ekstremt svært.

The Original text to the quote on Page 144: Der var jo den klare line fra Nybølle over Hald til Rasch, at det blev mere og mere teoretisk.

The Original text to the quote on Page 144: Rasch var virkelig en professor af den gamle skole. Han ytrede den opfattelse at "Hvis 5 procent af de studerende forstår hvad jeg mener, så er jeg fornøjet."

The Original text to the quote on Page 145: Rasch var en internationalt anerkendt, særdeles original statistiker. Men hans ideer og metoder passede dårligt til, hvad der var standard blandt økonomer. De almindeligt anvendte metoder baseret på normalfordelingen, f.eks. regressionsanalysen, så han særdeles kritisk på. Han afskaffede den gamle valgfrihed mellem stor og lille statistik, og indførte sin egen nye statistik. I hans tid var der en betydelig afstand mellem, hvad statistisk institut beskæftigede sig med, og hvad økonometrikerne på Økonomisk Institut arbejdede med.

The Original text to the quote on Page 147: Det som simpelthen karakteriserede Rasch's måde, det var simpelthen forkyndelse, indremissionsk, den fundamentalistiske stil, som er hentet fra indremission. Simpelthen den samme type, samme struktur som ved en forkyndelse. [...] Og så dette her med hvordan man maser sig ind på andre, og prøver at få dem ind i sin menighed.

The Original text to the quote on Page 148: Og de første mange uger gik jo med at fortælle om det her store møntforsøg, og det er jo sådan en empirisk tilgang til statistik [...] Det kan godt være at de der lange introducerende eksempel er gået hen over hovedet på de fleste.

The Original text to the quote on Page 148: Han viste stor respekt over for data. Data skulle man altså tage alvorligt. Og behandle dem på en tænksom måde og ikke på en kogebogsmåde. Man skulle ikke bare bruge færdigsyede metoder for det kunne være at data sagde noget andet. [...] Et synspunkt jeg kan huske, at han havde, og som jeg syntes var helt rigtigt, var at empirisk analyse, det var ikke bare kvantitativ statistisk analyse. Det var også case studies, hvor man prøvede at se på hvad er det egentlig for kausale virkninger, der gør sig gældende. Den statistiske analyse skal helst kombineres med sådan noget, og dermed helst med fagvidenskaben. Det var vigtigt at statistikeren havde en fag-videnskabsmand også når han arbejde.

The Original text to the quote on Page 149: Det var tidligere et alm. dogme, at fordelinger, hvis de ikke ligefrem var 'født' diskontinuerte (som f.eks. møntforsøg), burde følge en normal fordeling. [...] Man undersøgte sjældent, om normal-modellen nu også passede ordentligt på observationerne. [...] Alligevel gennemførte man en mangfoldighed af tests, der netop byggede på normaliteten, til en kontrol af andre specificerede hypoteser.
Denne indstilling er naturligvis kun tilladelig, dersom observationerne eller en transformation af dem kan beskrives ved Gauss-loven. Men dette vil meget hyppigt ikke være tilfældet, og er det ikke det, skal der kun mindre afvigelse til for at forringe slutningsgrundlaget under en række gængse anvendte tests.

The Original text to the quote on Page 150: Så i en eller anden forstand tror jeg trygt man kan sige, at det startede lidt som en katastrofe for forbindelsen mellem økonomerne og statistikken, fordi det, som økonomerne brugte var jo sådan set regressionsanalyse og normalfordelingen. Og Rasch var kritisk over for begge dele, og afskaffede begge dele.

The Original text to the quote on Page 152: Det var også sådan at han tænkte når han underviste. Han lirede det ikke bare af. Det var nærmest sådan at han forskede mens han underviste. Der var selvfølgelig også mange der stod af på det, og opfattede ham som upædagogisk, hvilket han måske også var på nogen måder. Men til gengæld var det jo spændende.

The Original text to the quote on Page 152: Jeg synes han var fantastisk inspirerende, altså den empiriske tilgang til det, hvor man virkelig skal trænge ned i materialet, før man bruger modeller. [...] Han var jo også vanvittig simpelthen. Han havde jo en fuldstændig vanvittig selvovervurdering af betydningen af de der målingsmodeller.

The Original text to the quote on Page 153: Rasch begyndte jo sin statistikundervisning med at afskaffe det foreliggende pensum, og han underviste første gang uden noget undervisningsmateriale, hvilket var vildt. Han fjernede undervisningen i varians- og regressionsanalyse og satte i stedet sin egen metode, hvor hvert datasæt skulle have sin skræddersyede stokastiske model, hvilket var fint, men ikke let at efterligne. Hector Estrup, som var den første der underviste i Johnstons Econometric Methods kunne ikke springe de indledende kapitler om regressionsanalyse over. Så statistikken og økonometrien i undervisningen gled fra hinanden.

The Original text to the quote on Page 153: Der var meget få økonomiske eksempler. Så hans eksempler havde ikke så meget pil til økonomerne. Det tror jeg var ret afgørende for at politstuderende ikke syntes det var så spændende. [...] Meget af det vi lærte i teoretisk statistik, mener jeg nu også jeg har glæde af som økonom. Stokastiske processer med videre. Men det var ikke økonomi og økonomiske data han arbejdede med, Rasch og hans ivrige kollegaer. Det kunne man godt mærke. Det med eksemplerne var utroligt vigtigt.

The Original text to the quote on Page 154: For vi lærte jo noget helt andet, end det vi følte, vi havde brug for i en eller anden forstand.

The Original text to the quote on Page 154: Så Rasch han måtte erkende, at vi andre syntes, at de traditionelle økonometriske metoder og statistiske indfaldsvinkler, som man jo bruger den dag i dag, må jeg jo skynde mig at sige, at det var det, der var værd at interessere sig for, og det kunne da godt være at Rasch havde

ret i nogen af sine kritikpunkter, men de metoder som han stillede til rådighed, og som jeg jo altså ikke er i stand til at vurdere kritisk selv overhovedet, de var i virkeligheden ikke særlig interessante.

The Original text to the quote on Page 154: Men der var ingen personlige uoverensstemmelser, bare intet samarbejde. Vi prøvede skam; Rasch og jeg havde et par møder tror jeg, hvor han tegne tidsserier på sit millimeterpapir- formentlig de lønserier du nævner - mens jeg forklarede om arbejdet med ADAM og specielt husker jeg at jeg forklarede, hvad forbrugsfunktioner var, og hvorfor jeg gik ud fra forbruget som bestemt af indkomsten. Det var Rasch ikke med på, han tjente de penge han skulle bruge og ikke omvendt. Problemet var jo at han havde sin - stokastiske og databaserede - model og jeg havde min på økonomisk teori og økonometrisk tradition baserede opskrift på en makromodel, og vi havde ikke videre brug for hinanden.

The original text to the quote of Page 154: Når man taler om Rasch og os andre, og jeg blev konsistoriet i 1962 så jeg oplevede hele perioden, men det interessante er, at vi havde så uendelig lidt meningsudviksling, og vi vidste som dårligt nok, hvordan Rasch han opfattede det. Men det lå ligesom i luften, at dem der interesserede sig for økonometri og den statistisk teori, der normalt doseres, de mistede ligesom kontakten med Rasch.

The Original text to the quote on Page 156: Og da spørgeskemaundersøgelse spiller en ret stor rolle i sociologi, og målingsmodellerne er designet til det, så var faget populært, i hvert fald for vores årgang af sociologistuderende, og vel også for den umiddelbart efterfølgende. Vores årgang var 1963 årgangen. Vi startede en 50 stykker.

The Original text to the quote on Page 156: Vi var mange der syntes det var sjovt. Blandt andet fordi, når vi nu kom der som sociologi-studerende, så var det sjovt at få noget at vide om noget mere kontant stof - hvor der var noget der var sandt, og noget der var falsk. Og hvor der kunne regnes. Og så der jo det gode ved det at vi samtidig blev konfronteret med eksempler fra psykologi, læseprøver og sådan. Og også lidt fra sociologi. [...] Og det der også morede nogen af os var, at han var så engageret selv. Og samtidig jo meget kritisk over for normalfordelingen.

The Original text to the quote on Page 157: Men så kom jo 68, og de yngre sociologistuderende, de begyndte at mene at statistik var alt for tungt at have med at gøre, og det var slet ikke nødvendigt. Og det blev anset for at være et svært fag blandt sociologer, det er der ingen tvivl om. Og der var nogen der gerne ville af med det. Så udviklingen blev, at på et eller andet tidspunkt fik sociologerne deres eget statistikkursus. Det har sikkert været i 69 eller 70. Udskilt fra politterne, og efterhånden blev det ret udvandet.

The Original text to the quote on Page 158: Og så tror jeg, at han havde en plan for sit institut; at nu skulle dette store gennembrud for videnskab. Målingsmodeller skulle udbredes til alle videnskaber, og derfor skulle der ansættes forskellige videnskabelige assistenter. Så Matthiesens opgave, det var demografi, og jeg blev ansat som økonomen, der skulle ændre økonomisk teori fuldstændigt til Rasch's principper, [...], men det var den der tanke at alle videnskaber skulle gennemsyres af målingsmodellerne, så derfor skulle han have en alsidig stabel fra forskellige områder.

The Original text to the quote on Page 159: Også som leder arbejdede han efter de store linier. Han interesserede sig ikke meget for praktiske detaljer, men forfulgte gerne en overordnet plan, ofte uanset planens realisme. Det er karakteristisk, at hans plan for Statistisk Institut's udbygning i 60'erne omfattede 4 selvstændige afdelinger og et tilsvarende stort antal medarbejdere. Selv i den periode, der nu betegnes 'de glade tressere', var denne plan urealistisk og nok en hemsko for et mere konkret samarbejde med økonomer og sociologer. Størst held havde han med etableringen af en demografisk afdeling, der genetablerede den nære forbindelse til befolkningsstatistikken, man havde haft i århundredets begyndelse.

The Original text to the quote on Page 159: Det der var typisk for Rasch, det var at han fik lavet grupper som arbejdet med meget forskellige faglige problemer. [...] Og det var også sådan der blev skaffet penge. Når man kan dokumentere at man har sådan en problemstilling og at man har en gruppe der arbejder med det så kommer der penge ind.

The Original text to the quote on Page 160: Han var simpelthen fantastisk til at formulere at der ikke forelå nogen resultater. Det fik han formuleret på en måde så han fik fornyet ansøgningen. [...] Han var suveræn på det punkt.

The Original text to the quote on Page 160: For det første synes jeg at han var en meget stor personlighed, og han var sådan en rigtig gammeldags professor som ikke længere findes. Han var en dominerende personlighed. Men han var meget large. [...] Rasch havde den der omsorg for sine medarbejdere. [...] Jeg kom jo så over på økonomisk institut. Der var meget få videnskabelige assistenter og få kandidatstipendiater. Der var Niels Thygesen og Erling Olsen, som var ansat som videnskabelige assistenter, og det var så meningen at de skulle skrive disputats. Men det her med sådan at snakke med folk og hjælpe til det her projekt, det lå de her gamle professorer helt fjern, for det var jo en opgave de havde, det der med at skrive disputats. [...] Men i hele det der selskab, der havde Rasch mere varme og den dominans som nogen beklager sig over, men han havde altså det der menneskelige format, så og largeness og sådanne nogle ting, så derfor er jeg en stor fan af Rasch.

The Original text to the quote on Page 161: Hvis man lavede noget skriftlig så gav Rasch nogle fantastiske kommentarer. [...] Han var virkelig grundig, og skrev en masse kommentarer i margen. Og når han så kom tilbage med det der, så blev man indkaldt hos Rasch. Han kunne være *voldsom*. Men altså, det var konstruktivt. Jeg følte mig ikke sur eller bitter. Jeg følte mig faktisk beriget. Men han var altså simpelthen så grov.

The Original text to the quote on Page 161: Jeg oplevede det miljø, der var på Sankt Peders stræde som et levende og dynamisk miljø. Hvor folk snakkede med hinanden om forskellige faglige problemer. Og de første år, hvor Rasch var der regelmæssigt, så havde man følelsen af at man havde en faglig enhed.

The Original text to the quote on Page 162: Det tør jeg ikke sige, men man kan sige, at han slap nemt om de administrative opgaver. Og han slap også relativt nemt om sin undervisning for han havde penge til at ansætte undervisningsassistenter til at holde forelæsninger for ham, som det egentlig var tanken at han selv skulle holde, for eksempel var min debut jo en sådan. [...] Rasch havde videnskabelige interesserer men ikke interesser i hvordan politstudiet skulle administreres - slet ikke. Det havde jeg indtrykket af, at han helst var fri for.

The Original text to the quote on Page 162 Så hans videnskab og hans udadvendte relationer, de kom frem ved internationale seminarer og ved den studiekreds, han havde blandt pædagoger. Der var jo mange Amerikanere, der kom hertil på besøg. Specielt amerikanere, der kom hertil, der arbejdede med testproblemer inden for psykologi.

The Original text to the quote on Page 162: Han var jo en excentrisk person på alle måder. Og på godt og ondt fulgte han de ting som han brændte for. Og han prøvede vel oprigtigt at udvikle noget som både økonomerne og sociologerne ville kunne bruge. Men så kom der jo et punkt. [...] Han fandt ud af, at han ikke rigtig kunne bidrage med noget væsentlig og så vendte han så tilbage til det, der havde været hans hovedopgave her i livet, nemlig at være konsulent. Han var en fremragende konsulent.

The Original text to the quote on Page 164: Rasch snakkede jo altid om at der skulle laves nogle lærebøger. Men det blev aldrig til nogen bøger fra ham. Derfor besluttede vi, en gruppe der bestod af Rasch Ulf Christiansen, Matthiesen og jeg, at vi skulle redigere den her lærebog.

The Original text to the quote on Page 164: Vi diskuterede det vældig meget de første kapitler, men så senere tabte Rasch interessen, og rejste væk. Han tog til Chicago.

The Original text to the quote on Page 164: Der skete jo også det at efter cirka 68 så kom han jo ikke på instituttet. Han var på Læsø. Men så cirka hver tredje måned kom han og var her nogle dage, men ellers var han på Læsø og så kunne vi ringe til ham der. [...] Han havde sygeligheder, han drak jo som en svamp. Så var det også sådan, at han faldt ned fra trappen på færgen til Læsø. Han havde nogle sygeligheder som han brugte som argumentation for at han ikke skulle komme ind på instituttet.

The Original text to the quote on Page 165: Og forskningsrådene var meget large over for ham. På socialforskningsinstituttet, Henning Friis, gjorde meget for ham. Og det var egentlig pudsigt, for var der en, Rasch ikke kunne fordrage, så var det Henning Friis. Han mente dog, at det var rigtigt, at den mand skulle støttes.

23 References

Andersen, Ellen (1975). *En Model for Danmark.* Akademisk forlag. København.

Andersen, Erling, B. (1980). Georg Rasch. Obituary in *Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1980-1981.* pp. 536−540.

Andersen, Erling, B. (1986). Georg Rasch. In Samuel Kotz and Norman L. Johnson (Eds.) *Kotz-Johnson. Encyclopedia of Statistical Sciences, Volume 7.* John Wiley and Sons, Inc. Canada. p. 626−627.

Andersen, Erling, B. (1995).What Georg Rasch would have Thought about this Book. In Gerhard H. Fischer, Ivo W. Molenaar (Eds.), *Rasch Models.* Springer-Verlag, New York, 383−390.

Andersen, Erling and Olsen, Lina Wøhlk (2001). The life of Georg Rasch as a Mathematician and as a Statistician. In Anne Boomsma, Marijtje A. J. van Duijn, Tom A.B.Snijders (Eds.).*Essays on Item Response Theory.* Lecture Notes in Statistics. Springer-Verlag New York, Inc.

Andrich, D. (2002). Understanding Rasch Measurement: Understanding Resistance to the Data-Model Relationship in Rasch's Paradigm: A reflection for the Next Generation. *Journal of Applied Measurement*, Vol 3, n. 3.

Kærgård, Niels (2001). 1960-1979 og årene efter: vækst, specialisering of formalis-

ering. p. 159-185. In Ditlev Tamm and Ejvind Slottved (Eds.) *Københavns Universitet 1979-1979, Bind VI.2. Det rets- og statsvidenskabelige fakultet 2. del.*, København.

Hald, Anders (1952). *Statistical Theory with Engineering Applications.* John Wiley and Sons, New York. Chapman and Hall, London.

Hald, Anders (1983). Statistikkens teori. In Mogens Phil and others (Eds.) *Københavns Universitet 1479-1979, bind XII: Det matematisk-naturvidenskabelige Fakultet, 1. del.*, G.E.C. Gads Forlag, København, pp. 213-228.

Hald, Anders (1994) *The Three Revolutions in Statistical Inference and Their Importance for my Life as a Statistician.* In: Three Contributions to the History of Statistics. Institute of Mathematical Statistics University of Copenhagen. Preprint 1994.

IMSOR (1971). Professor, dr. phil. Georg Rasch. Oversigt over væsentligere arbejder. Udgivet i anledning af hans 70 års dag. 21. 09. 71). Published by IMSOR. (IMSOR – Institut for Matematisk Statistik og Operationsanalyse).

Jørgensen, F. Abildgård (1957). *Kompendium af statistik for psykologer og pædagoger.* København.

Kerrich, J.E. (1950). *An Experimental Introduction to the Theory of Probability.* Denmark.

Mossin, Axel (1972). A Mean Demand Function and Individual Demand Functions Confronted with the Weak and the Strong Axioms of Revealed Preference: An Empirical Test. *Econometrica, Vol. 40,* No.1, p. 177-192.

Neter, J. and Wasserman, W. (1961). *Fundamental Statistics for Business and Economics.* Allyn and Bacon, Inc. Boston.

Nielsen, N. and Rasch, G. (1923). Notes Supplémentaires sur les Equations de Lagrange. *Det Kgl. Danske Videnskabernes Selskab. Matematisk-fysiske Meddelelser. V, 7.* Copenhagen.

Ramskov, Kurt (1995). *Matematikeren Harald Bohr.* Institut for de eksakte videnskabers historie. Det naturvidenskabelige Fakultet. Aarhus Universitet.

Rasch, G. (1930). *Om Matrixregning og dens Anvendelser paa Differens- og Differentialligninger.* Copenhagen: Levin and Munksgaars Forlag.

Rasch, Georg (1953). On Simultaneous Factor Analysis in Several Populations. Uppsala Symposium on Psychological Factor Analysis. 17-19 March 1953. *Nordisk Psykologi's Monograph, Series No. 3*, p. 65-71.

Rasch, Georg (1960). *Probabilistic Models for Some Intelligence and Attainment Tests.* Copenhagen: The Danish Institute of Educational Research.

Rasch, Georg (1961). On General Laws and the Meaning of measurement in Psychology. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Theory of Probability,* Vol. IV, p. 321-333, Berkeley.

Rasch, Georg (1966). *Referat af professor G. Rasch's forelæsninger over statistikkens teori.* Udarbejdet af Ulf Christiansen. Københavns Universitets fond til tilvejebringelse af læremidler. København.

Rasch, Georg (1968). *G. Rasch's lærebog i statistik, bind 1.*, Skrevet af Ulf Christiansen og Jon Stene. Teknisk Forlag. København.

Rasch, G. (1968'). *A Mathematical Theory of Objectivity and Its Consequences for Model Construction.* Paper presented at the European Meeting on Statistics, Econometrics and Management Sciences. Amsterdam.

Rasch, Georg (1969). *G. Rasch's lærebog i statistik, bind 2.* Skrevet af Ulf Christiansen og Jon Stene. København.

Rasch, G. (1972). Objektivitet i Samfundsvidenskaberne et metodeproblem. *Nationaløkonomisk tidsskrift,* 110, 161−196.

Rasch, Georg (1977). On Specific Objectivity. An Attempt at Formalizing the Request for Generality and Validity of Scientific Statements. *Danish Yearbook of Philosophy,* Vol 14, p. 58-94.

Weibull, Christer (1970). Book Review of G. Rasch's lærebog i teoretisk statistik af Ulf Christiansen og Jon Stene. *Nationaløkonomisk Tidsskrift,* Vol 108, p. 346-348.

Wolf, Torben (2001). Sociologi. *Københavns Universitet 1979-1979, Bind VI.2. Det rets- og statsvidenskabelige fakultet 2. del.* Ed. Ditlev Tamm og Ejvind Slottved. København.

Wright, Benjamin (1980). Foreword. In G. Rasch, *Probabilistic models for some intelligence and attainment tests.* Copenhagen: DPI (Expanded edition, 1980, Chicago: The University of Chicago Press. (pp. ix−xix).

Årbog (1958–1963). Årbog for Københavns Universitet, Indeholdende Meddelelser for de akademiske År 1958-1963. København.

Årbog (1963–1964). Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1963-1964. København.

Årbog (1964–1965). Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1964-1965. København.

Årbog (1965–1966). Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1965-1966. København.

Årbog (1966–1967). Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1966-1967. København.

Årbog (1967–1968). Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1967-1968. København.

Årbog (1968–1969). Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1968-1969. København.

Årbog (1969–1970). Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1969-1970. København.

Årbog (1970–1971). Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1970-1971. København.

Årbog (1971–1972). Årbog for Københavns Universitet, Indeholdende Meddelelser for det akademiske År 1971-1972. København.

Exact Rasch Testing

Lina Wøhlk Olsen[1]

Chapter 5

ABSTRACT This article introduces the exact Rasch test approach for test of fit of some discrete log linear models. Like in Fisher's exact test of a two by two contingency table the conditional probabilities are used to calculate the level of significance. In some cases, however, these conditional probabilities provide a unique identification of the model. This implies that a set of observations is extreme under the model *if and only if* a small level of significance is obtained.
KEY WORDS: Goodness of fit, Fisher's exact test, contingency tables, log linear models.

1 Introduction

The object of this article is to combine the exact conditional test approach with Georg Rasch's work on specific testing (Rasch, 1974). The resulting combination is denoted *exact Rasch testing*.

In Rasch's terminology a specific test is based on a probabilistic statement, which is a necessary and sufficient condition for the hypothesis. Some log-linear models are uniquely identified by their conditional probabilities given the sufficient statistics. A test based on such conditional probabilities is, by definition, specific. Since exact conditional testing uses the conditional probabilities to calculate the significance level, it is straightforward to combine specific testing and exact conditional testing in these cases.

The exact Rasch test defines the test statistic as the random counterpart to the conditional probabilities. This choice is justified since these conditional probabilities provide a unique identification of the model. This implies that an observation is extreme under the model if and only if the observed value of the test statistic is extreme. That is, if the exact test gives a small significance level.

Section 2 gives an introduction to exact Rasch testing for the case of goodness

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of fit tests. In section 3 it is demonstrated that a version of Fisher's exact test is also an exact Rasch test. Section 4 demonstrates that a class of discrete log-linear models are uniquely identified by their conditional probabilities given the sufficient statistics. That is, that the exact Rasch test approach is applicable for these models. In Appendix A, an exact Rasch test of Rasch's multi dimensional Rasch model with multi dimensional parameters is derived.

2 Exact Rasch Testing

The exact Rasch test approach will be illustrated through an example. We shall see how the approach can be used to examine whether a data set fits a multiplicative Poisson model.

Let X_{ij}, $i = 1, ..., I$, $j = 1, ..., J$ be independent random variables, and let $[X_{ij}]$ be a short hand notation for these variables collected in a matrix:

$$[X_{ij}] = \begin{pmatrix} X_{11} & \cdots & X_{1J} \\ \vdots & \ddots & \vdots \\ X_{I1} & \cdots & X_{IJ} \end{pmatrix}.$$

Let R and C be vectors that denote the row and column marginals of $[X_{ij}]$, that is $R = (X_{1.}, ..., X_{I.})^t$ respectively $C = (X_{.1}, ..., X_{.J})^t$. Throughout this article the corresponding lower-case letters denotes observations of random variables. The multiplicative Poisson model is given by:

$$p_{\theta,\sigma}([x_{ij}]) = \prod_{i=1}^{I} \prod_{j=1}^{J} e^{-\theta_i \sigma_j} \frac{(\theta_i \sigma_j)^{x_{ij}}}{x_{ij}!} \tag{5.1}$$

where $\theta = (\theta_1, ..., \theta_I) \in \mathbf{R}^I$ and $\sigma = (\sigma_1, ..., \sigma_J) \in \mathbf{R}^J$ are unknown parameters. It has been shown that the multiplicative Poisson model (5.1) is equivalent with the following probabilistic statement (Rasch, 1974), (Gottschau, 1983):

$$p([x_{ij}] \mid r, c) = \frac{\prod_{i=1}^{I} x_{.i}! \prod_{j=1}^{J} x_{j.}!}{x_{..}! \prod_{i=1}^{I} \prod_{j=1}^{J} x_{ij}!}. \tag{5.2}$$

Thus, the conditional probabilities given the sufficient statistics provide a unique identification of the model (5.1).

The conditional probabilities (5.2) will be used to test the goodness of fit of the multiplicative Poisson model. The present choice of test statistic is similar to that

of Carr's (1980), namely the random counterpart to the conditional probabilities,

$$q([X_{ij}]) = \frac{\prod_{i=1}^{I} X_{.i}! \prod_{j=1}^{J} X_{j.}!}{X_{..}! \prod_{i=1}^{I} \prod_{j=1}^{J} X_{ij}!}.$$

Small values of q are extreme. The test statistic, q, has quite an appealing interpretation. When observing $[X_{ij}] = x_0$ with marginals r_0 and c_0, an observation, say $\tilde{x}$, *is more extreme than* x_0 *if and only if* $\tilde{x}$ *has less probability of being observed than* x_0, conditional on r_0 and c_0.

For the test statistic, q, the significance level equals the probability of observing something that has less probability of being observed than x_0, conditional on r_0 and c_0,

$$\epsilon(x_0) = P\left(q([X_{ij}]) \leq q(x_0) \mid R = r_0\ C = c_0\right)$$

$$= \sum_{[x_{ij}] \in \Omega(r_0, c_0)} 1_{\{q([x_{ij}]) \leq q(x_0)\}} \frac{\prod_{i=1}^{I} x_{.i}! \prod_{j=1}^{J} x_{j.}!}{x_{..}! \prod_{i=1}^{I} \prod_{j=1}^{J} x_{ij}!},$$

where $\Omega(r_0, c_0)$ denotes the set of outcomes, $[x_{ij}]$, with row and column marginals r_0 and c_0.

Now, this combination of Rasch's specific testing with exact conditional testing allows for a quite unique interpretation of the test; because the conditional probabilities (5.2) provide a unique identification of the model (5.1), *an observed value of the test statistic,* $q(x_0)$*, is extreme in the distribution of* q *if and only if* x_o *is extreme as distributed according to the multiplicative Poisson model (5.1).* But this is equivalent to obtaining a small significance level. For further references we shall call this combination of Rasch's specific testing and exact conditional testing for *exact Rasch testing.* In general, an exact Rasch test may be defined as follows:

Definition. Let X be a p dimensional discrete random variable, distributed according to $f_\theta(x)$, where $\theta \in \mathbf{R}^k$ is a parameter. Let $S = h(X)$ be a function of X, and let x_0 and $s_0 = h(x_0)$ be observations. Assume that $P(X \mid S)$ is independent of all parameters and that $P(X \mid S)$ is not a uniform distribution. If $P(X \mid S)$ provides a unique identification of the distribution of X, we shall say that an exact Rasch test is conducted when $P(X \mid S)$ is used as test statistic, and the probability of significance is calculated as:

$$\epsilon(x_0) = \textstyle\sum_{x \in \Omega(s_0)} 1_{\{p(x|s_0) \leq p(x_0|s_0)\}} p(x \mid s_0),$$

where $\Omega(s_0)$ denotes the set of outcomes with $S = s_0$.

Several authors have raised doubts about ordering the sample solely on the basis of probability of occurrence, as is done when performing an exact Rasch test, see Radlow and Alf (1975). The rationale behind the criticism is that some outcomes may be less likely than the observation, but still, their discrepancies from the null hypothesis measured by some test statistics, for instance the χ^2, are smaller than that of the actual observation. As a reaction to this criticism, well known and thoroughly investigated test statistics such as the χ^2 or the likelihood ratio statistic are preferred when making exact conditional tests, see for instance Agresti and Wackerly (1977). Here, however, it is argued that when it is possible to make an exact Rasch test, the probability of occurrence ought to be reconsidered as a test statistic; the point being that the exact Rasch test is the only test approach for which it possible to claim that an observation is extreme as distributed according to the model *if and only if* a small level of significance is obtained.

3 Test of Parametric Structures

The exact Rasch test approach also applies to hypothesis about the parametric structure of a statistical model. As an example we shall see that Fisher's exact test of a two by two contingency table is, in a sense, an exact Rasch test (Rasch, 2001).

Assume that X_1 and X_2 are independent random variables, each distributed according to a binomial: $X_i \sim bin(n_i, p_i) \quad i = 1, 2$. The null hypothesis is

$$H_0 : \quad p_1 = p_2.$$

To make an exact Rasch test of H_0 we have to find a conditional distribution that gives a unique identification of H_0.

Writing $\theta_i = p_i/(1 - p_i) \quad i = 1, 2$ the binomials have the form

$$p_{\theta_i}(x_i) = \binom{n_i}{x_i} \frac{\theta_i^{x_i}}{(1+\theta_i)^{n_i}} \quad i = 1, 2.$$

Let $Z = X_1 + X_2$, and let $\Omega(z) = \{(x_1, x_2) \mid x_1 + x_2 = z\}$. We now have that

$$p_{\theta_1,\theta_2}(z) = \frac{\sum_{(\tilde{x_1},\tilde{x_2})\in\Omega(z)} \binom{n_1}{\tilde{x}_1}\theta_1^{\tilde{x}_1} \binom{n_2}{\tilde{x}_2}\theta_2^{\tilde{x}_2}}{(1+\theta_1)^{n_1}(1+\theta_2)^{n_2}}$$

which implies that

$$p_{\theta_1,\theta_2}(x_1,x_2 \mid z) = \frac{\binom{n_1}{x_1}\theta_1^{x_1}\binom{n_2}{x_2}\theta_2^{x_2}}{\sum_{(\tilde{x_1},\tilde{x_2})\in\Omega(z)}\binom{n_1}{\tilde{x}_1}\theta_1^{\tilde{x}_1}\binom{n_2}{\tilde{x}_2}\theta_2^{\tilde{x}_2}}. \tag{5.3}$$

Under H_0 equation (5.3) simplifies to

$$p(x_1,x_2 \mid z) = \frac{\binom{n_1}{x_1}\binom{n_2}{x_2}}{\binom{n_1+n_2}{z}}. \tag{5.4}$$

Consequently, (5.4) is a necessary condition for H_0. To show that it is also a *sufficient condition*, identify equation (5.4) and (5.3)

$$\sum_{(x_1,x_2)\in\Omega(z)}\binom{n_1}{x_1}\theta_1^{x_1}\binom{n_2}{x_2}\theta_2^{x_2} = \binom{n_1+n_2}{z}\theta_1^{x_1}\theta_2^{x_2}. \tag{5.5}$$

Equation (5.5) is valid for all x_1 and x_2 with sum z. The two equations obtained from $(x_1,x_2)=(z,0)$ respectively $(x_1,x_2)=(0,z)$ gives the identity $\theta_1=\theta_2$.

In consequence, if $X_i \sim bin(n_i,p_i) \quad i=1,2$, then equation (5.4) is a necessary and sufficient condition for H_0. This implies that an observation $x_0=(x_{1_0},x_{2_0})$ is extreme under H_0 if and only if the observed value of the test statistic $q(X_1,X_2) = \binom{n_1}{X_1}\binom{n_2}{X_2}/\binom{n_1+n_2}{Z}$ is extreme, that is, if the significance level,

$$\epsilon(x_0) = \sum_{(x_1,x_2)\in\Omega(z_0)} 1_{\{q(x_1,x_2)\le q(x_{1_0},x_{2_0})\}}\frac{\binom{n_1}{x_1}\binom{n_2}{x_2}}{\binom{n_1+n_2}{x_1+x_2}},$$

is small.

It is thus possible to generalize the exact Rasch test approach to tests of parametric structures. No further details will be given on this subject, however, the point being that the exact Rasch test approach does not test against a specific hypothesis. When testing goodness of fit of a model this does not seem to be a problem. In some cases it might even be erroneous to decide on a specific model as alternative, in effect limiting the attention to a special class of models. But when testing hypotheses about the parametric structure of a model, often the alternative hypothesis is of main interest, and in such cases, lack of an alternative hypothesis seems to be a serious problem.

4 Exact Rasch tests in m dimensional contingency tables

As we shall see the exact Rasch test approach is applicable for testing various discrete statistical models in $m \ge 2$ dimensional contingency tables. The general case

is demonstrated using Gottschau's result for a two dimensional contingency table:[2]

Theorem 1 (Gottschau, 1983, p. 115-116)
Let $[X_{ij}]$ be a matrix of locally independent discrete random variables with row and column marginals R and C. Let $\theta = (\theta_1, ..., \theta_I)$ and $\sigma = (\sigma_1, ..., \sigma_J)$ be unknown parameters. Let $(h_{ij})_{i=1,...,I\ j=1,...,J}$ be functions of the observations only and let $(\phi_{ij})_{i=1,...,I\ j=1,...,J}$ be functions of the parameters only. The three probabilistic statements, (5.6), (5.7) and (5.8) are equivalent.

$$p_{\theta,\sigma}([x_{ij}]) = \prod_{i,j} \frac{1}{\phi_{ij}(\theta_i, \sigma_j)} e^{(\theta_i x_{ij} + \sigma_j x_{ij})} h_{ij}(x_{ij}) \tag{5.6}$$

$$p_{\sigma}([x_{ij}] \mid r) = \prod_{i=1}^{I} \frac{\prod_{j=1}^{J} e^{\sigma_j x_{ij}} h_{ij}(x_{ij})}{\sum_{\{\tilde{[x_{ij}]} | \tilde{r}=r\}} \prod_{j=1}^{J} e^{\sigma_j \tilde{x}_{ij}} h_{ij}(\tilde{x}_{ij})} \tag{5.7}$$

$$p([x_{ij}] \mid r, c) = \frac{\prod_{i=1}^{I} \prod_{j=1}^{J} h_{ij}(x_{ij})}{\sum_{\{\tilde{[x_{ij}]} | \tilde{r}=r, \tilde{c}=c\}} \prod_{i=1}^{I} \prod_{j=1}^{J} h_{ij}(\tilde{x}_{ij})}. \tag{5.8}$$

The generalization to higher dimensions involves rather tedious notation. Essentially, a log-linear model with marginals of the m dimensional table as sufficient statistics is introduced. Interaction terms are allowed. To show that this model is uniquely identified by it's conditional probabilities given the sufficient statistics, two restrictions are imposed on the model. This is done to make the model look like the models of Theorem 1. The first restriction is, loosely speaking, that each dimension of the table must be accounted for. This can be done for instance by parameterizing the dimensions as in model (5.6), by conditioning as in model (5.8), or by a combination of both as in model (5.7). The second restriction is that no categorical variable included in the model must be nested in the categorical variable generated by the interaction of the remaining categorical variables. These two restrictions make it possible to make an easy proof. The model, the two restrictions and the theorem will now be stated.

Let $X_{i_1,...,i_m}$, $i_1 = 1, ..., I_1$,..., $i_m = 1, ..., I_m$ be independent discrete random variables in a m dimensional contingency table, and let $[X_{i_1,...,i_m}]$ be a short hand notation for the collection of theses variables.

[2] A proof of Theorem 1 is given in Appendix B.

Let M_s be a non empty subset of $\{1, ..., m\}$, $M_s \neq \{1, ..., m\}$, $s = 1, ..., w$, and let j_s be a super index that corresponds to those indexes i_l for which $l \in M_s$. The marginals of the contingency table may be written

$$t^s_{j_s} = \sum_{i_l : l \in \{1,...,m\} \setminus M_s} x_{i_1,...,i_m}.$$

To ease the notation, let for each $s \in \{1, ..., w\}$ R_s be a vector that contains the marginals $t^s_{j_s}$, that is, $R_s = (t^s_1, ..., t^s_{J_s})^t$, where $J_s = \prod_{l \in M_s} J_l$.

The parameters are written as follows: Assume each dimension, p, $p = 1, ..., m$, of the table is described by a categorical variable, F^p, on I_p levels. Let $\{\lambda^{M_s}_{j_s} \mid j_s = 1, ..., J_s\}$ be a set of parameters which correspond to the interaction between the categorical variables, F^p, for which $p \in M_s$. Let Λ_w be a short hand notation for all parameters, that is, $\Lambda_w = \{\lambda^{M_s}_{j_s} \mid j_s = 1, ..., J_s \ s = 1, ..., w\}$.

Further, let $h_{i_1,...,i_m}$ be functions of the observations only, and let $\psi_{i_1,...,i_m}$ be functions of the parameters only, $i_1 = 1, ..., I_1$,..., $i_m = 1, ..., I_m$.

We shall consider the following model

$$f_{\Lambda_w}([x_{i_1,...,i_m}]) = \exp(\sum_{s=1}^{w} \sum_{j_s=1}^{J_s} t^s_{j_s} \lambda^{M_s}_{j_s}) \prod_{i_1,...,i_m} h_{i_1,...,i_m}(x_{i_1,...,i_m}) \psi_{i_1,...,i_m}(\Lambda_w). \quad (5.9)$$

Without loss of generality it can be assumed that the model (5.9) is parameterized by the highest order of interaction between the categorical variables $F^1, ..., F^m$. For instance, if the model includes a parameter for the interaction between F_1 and F_2, it is assumed that the model does not include parameters for the main effects F_1 and F_2 as well.

To generalize Theorem 1 two assumptions are needed, namely $A1$ and $A2$.

$A1$. It is assumed that each dimension of the m dimensional contingency table is accounted for in model (5.9). Technically, this imply that for each index $i_\nu \in \{i_1, ..., i_m\}$, the model (5.9) must either contain a set of parameters $\{\lambda^{M_s}_{j_s} \mid j_s = 1, ..., J_s\} \subseteq \Lambda_w$ such that $\nu \in M_s$, or the probability space for the model (5.9) must be characterized by that there exist a known marginal of the table, say

$t^s_{j_s} = \sum_{i_l : l \in \{1,...,m\} \setminus M_s} x_{i_1,...,i_m}$, such that $\nu \in M_s$.

*A*2. It is assumed that no matter which subset, B_s, of $\{\{1,...,w\} \setminus s\}$ that is chosen, the intersection between $\cup_{s' \in B_s} M_{s'}$ and M_s will not include M_s, that is

$$\forall B_s,\ s \in \{1,...,w\} : \quad \cup_{s' \in B_s} M_{s'} \cap M_s \neq M_s.$$

Theorem 2

For model (5.9) with restriction *A*1 and *A*2 the distribution of

$$([x_{i_1,...,i_m}] \mid R_1, ..., R_w)$$

provides a unique identification of the model (5.9).

Proof

Theorem 1 states that Theorem 2 is valid for $w = 2$. Assume that Theorem 2 is valid for $w = d - 1 > 2$, $d \in \mathbf{N}_0$. We shall reparameterize the model

$$f_{\Lambda_d}([x_{i_1,...,i_m}]) = \exp(\sum_{s=1}^{d} \sum_{j_s} t^s_{j_s} \lambda^{M_s}_{j_s}) \prod_{i_1,...,i_m} h_{i_1,...,i_m}(x_{i_1,...,i_m}) \psi_{i_1,...,i_m}(\Lambda_d). \tag{5.10}$$

Therefore, let $\sigma_j = \lambda^{M_d}_{j_d}$, $j = j_d$ and let $\theta_\nu = \sum_{s=1}^{d-1} \lambda^{M_s}_{j_s}$, where ν is the index corresponding to those indexes, i_l, for which $l \in \{1,...,m\} \setminus M_d$. In this parameterization the sufficient statistics are, respectively, t^d_j and

$$t_\nu = \sum_{i_l : l \in M_d} x_{i_1,...,i_m}.$$

Writing $x_{i_1,...,i_m} = x_{\nu,j}$ the model (5.10) may be written

$$f_{\sigma,\theta}([x_{\nu,j}]) = \exp(\sum_{\nu} \theta_\nu t_\nu + \sum_{j} \sigma_j t^w_j) \prod_{\nu j} h_{\nu,j}(x_{\nu,j}) \psi_{\nu,j}(\theta_\nu, \sigma_j). \tag{5.11}$$

Due to the assumption *A*2 the set of categorical variables which determines the σ_i's are not a subset of the set of categorical variables which determines the θ_ν's. This, together with assumption *A*1, implies that the model (5.11) is of the same type as the model (5.6) of Theorem 1. Now, according to Theorem 1, equation (5.11) is equivalent to

$$f_\theta([x_{\nu,j}] \mid R_d) = \prod_j \left(\frac{\prod_\nu \exp(\theta_\nu x_{\nu,j}) h_{\nu,j}(x_{\nu,j})}{\sum_{\{\tilde{[x_{\nu,j}]} \mid \tilde{R}_d = R_d\}} \prod_\nu \exp(\theta_\nu \tilde{x}_{\nu,j}) h_{\nu,j}(\tilde{x}_{\nu,j})} \right). \tag{5.12}$$

In the original parameterization (5.12) equals

$$f_{\Lambda_{d-1}}([x_{\nu,j}] \mid R_d) = \frac{\prod_{i_1,...,i_m} \exp(\sum_{s=1}^{d-1} \lambda_{j_s}^{M_s} x_{i_1,...,i_m}) h_{i_1,...,i_m}(x_{i_1,...,i_m})}{\prod_j \sum_{\{[\tilde{x_{\nu,j}}]|\tilde{R}_d=R_d\}} \prod_\nu \exp(\theta_\nu \tilde{x}_{\nu,j}) h_{\nu,j}(\tilde{x}_{\nu,j})}. \quad (5.13)$$

For R_d given, the denominator is independent of the $x_{i_1,...,i_m}$'s. We can therefore rewrite (5.13) as

$$f_{\Lambda_{d-1}}([x_{\nu,j}] \mid r_d) = \exp(\sum_{s=1}^{d-1} \sum_{j_s} t_{j_s}^s \lambda_{j_s}^{M_s}) \prod_{i_1,...,i_m} h_{i_1,...,i_m}(x_{i_1,...,i_m}) \tilde{\psi}_{i_1,...,i_m}(\Lambda_{d-1}). \quad (5.14)$$

Obviously, the model (5.14) meets the requirements $A1$ and $A2$. Therefore, according to the assumption that Theorem 2 is valid for $w = d-1$, a unique identification of the model (5.14) is obtained from (5.14) by conditioning on $R_1, ..., R_{d-1}$. In consequence the conditional distribution of $[X_{i_1,...,i_m}]$ given $R_1, ..., R_d$ provides a unique identification of the model (5.10).

Example

As an example of the usefulness of Theorem 2, we shall make exact Rasch tests of the fit of a discrete log-linear model in a four dimensional contingency tables. In the notation introduced in the beginning of this section we shall consider models with interaction between two of the categorical variables, say F_1 and F_2, and main effects of the remaining two variables, F_3 and F_4. The model may be written

$$\begin{aligned} f_{\Lambda_3}([x_{i_1,i_2,i_3,i_4}]) = \exp(\textstyle\sum_{i_1,i_2} x_{i_1,i_2,.,.} \lambda_{i_1,i_2}^{\{1,2\}}) \exp(\sum_{i_3} x_{.,.,i_3,.} \lambda_{i_3}^{\{3\}} + \sum_{i_4} x_{.,.,.,i_4} \lambda_{i_4}^{\{4\}}) \\ \times \textstyle\prod_{i_1,i_2,i_3,i_4} h_{i_1,i_2,i_3,i_4}(x_{i_1,i_2,i_3,i_4}) \psi_{i_1,i_2,i_3,i_4}(\Lambda_3). \end{aligned} \quad (5.15)$$

The choices of the h_{i_1,i_2,i_3,i_4}'s and ψ_{i_1,i_2,i_3,i_4}'s determines the model. For example, if we choose $h_{i_1,i_2,i_3,i_4}(x_{i_1,i_2,i_3,i_4}) = x_{i_1,i_2,i_3,i_4}!$ and $\psi_{i_1,i_2,i_3,i_4}(\Lambda_3) = \exp(\exp(\lambda_{i_1,i_2}^{\{1,2\}} + \lambda_{i_3}^{\{3\}} + \lambda_{i_4}^{\{4\}}))$ the model (5.15) is a Poisson.

According to Theorem 2 the model (5.15) is uniquely identified by the conditional probabilities given the sufficient statistics. The sufficient statistics are

$$\begin{aligned} R_1 &= (X_{1,1,.,.}, ..., X_{I_1,I_2,.,.})^t \\ R_2 &= (X_{.,.,1,.}, ..., X_{.,.,I_3,.})^t \\ R_3 &= (X_{.,.,.,1}, ..., X_{.,.,.,I_4})^t, \end{aligned}$$

and the conditional probabilities given the sufficient statistics are

$$p([x_{i_1,i_2,i_3,i_4}] \mid r_1, r_2, r_3) = \frac{\prod_{i_1,i_2,i_3,i_4} h_{i_1,i_2,i_3,i_4}(x_{i_1,i_2,i_3,i_4})}{\sum_{\tilde{x}\in\Omega(r_1,r_2,r_3)} \prod_{i_1,i_2,i_3,i_4} h_{i_1,i_2,i_3,i_4}(\tilde{x}_{i_1,i_2,i_3,i_4})},$$

where $\Omega(r_1, r_2, r_3) = \{\tilde{x} \mid \tilde{r_1} = r_1, \tilde{r_2} = r_2, \tilde{r_3} = r_3\}$. Therefore, to make an exact Rasch test, choose as test statistic the random variable

$$q([X_{i_1,i_2,i_3,i_4}]) = \frac{\prod_{i_1,i_2,i_3,i_4} h_{i_1,i_2,i_3,i_4}(X_{i_1,i_2,i_3,i_4})}{\sum_{\tilde{x}\in\Omega(R_1,R_2,R_3)} \prod_{i_1,i_2,i_3,i_4} h_{i_1,i_2,i_3,i_4}(\tilde{x}_{i_1,i_2,i_3,i_4})}, \tag{5.16}$$

and, when observing x_0, calculate the level of significance as

$$P\left(q([X_{i_1,i_2,i_3,i_4}]) \leq q(x_0) \mid R_1 = r_{1_0}\ R_2 = r_{2_0}\ R_3 = r_{3_0}\right) =$$

$$\frac{\sum_{x\in\Omega(r_{1_0},r_{2_0},r_{3_0})} 1_{\{q(x)\leq q(x_0)\}} \prod_{i_1,i_2,i_3,i_4} h_{i_1,i_2,i_3,i_4}(x_{i_1,i_2,i_3,i_4})}{\sum_{\tilde{x}\in\Omega(r_{1_0},r_{2_0},r_{3_0})} \prod_{i_1,i_2,i_3,i_4} h_{i_1,i_2,i_3,i_4}(\tilde{x}_{i_1,i_2,i_3,i_4})}.$$

Now, *if, and only if*, the significance level is small, the observation, x_0, is extreme as distributed according to the model (5.15).

Notice the importance of specifying the sample space when making an exact Rasch test. For example, if the h_{i_1,i_2,i_3,i_4}'s and ψ_{i_1,i_2,i_3,i_4}'s are chosen such that the model (5.15) is a multinomial with the total number of observations, n, known

$$p_{\Lambda_3}([x_{i_1,i_2,i_3,i_4}]) = n! \prod_{i_1,i_2,i_3,i_4} \frac{e^{(\lambda_{i_1,i_2}^{\{1,2\}} + \lambda_{i_3}^{\{3\}} + \lambda_{i_4}^{\{4\}}) x_{i_1,i_2,i_3,i_4}}}{x_{i_1,i_2,i_3,i_4}!} \tag{5.17}$$

the test statistic becomes

$$q(X) = \frac{\prod_{i_1,i_2,i_3,i_4} X_{i_1,i_2,i_3,i_4}!}{\sum_{\tilde{x}\in\Omega(R_1,R_2,R_3)} \prod_{i_1,i_2,i_3,i_4} \tilde{x}_{i_1,i_2,i_3,i_4}!}.$$

Inserting the h_{i_1,i_2,i_3,i_4} functions of a Poisson, namely $h_{i_1,i_2,i_3,i_4}(x_{i_1,i_2,i_3,i_4}) = x_{i_1,i_2,i_3,i_4}!$, in equation (5.16) shows, that the same test statistic is used to test the goodness of fit of a Poisson and the multinomial (5.17). However, the sample space distinguishes between the two models.

5 Summary

This article has introduced the exact Rasch test approach. Essentially, the approach is a combination of making exact tests, and of utilizing that in some cases the

conditional probabilities used to make the exact tests, do in fact provide a unique identification of the model. The implication of making an exact Rasch test is quite remarkable, namely that *if and only if* the significance level is small, a set of observations is extreme as distributed according to the model.

The exact Rasch test approach is demonstrated accessible for discrete log linear models in $m > 1$ dimensional contingency tables, though with certain limitations, the most important being that the sufficient statistics of the model are bound to be marginals of the table.

6 Appendix A. An exact Rasch test of the multidimensional Rasch model

In this appendix an exact Rasch test of Rasch's multidimensional Rasch model with multi dimensional parameters will be derived.

Rasch is probably best known for the dichotomous Rasch model; a model which is widely used in the analysis of attainment tests and questionnaires. The model is aimed at a situation with two possible responses, such as right/wrong, or agree/disagree, that is, the sample space may be denoted by $x_{ij} \in \{0, 1\}$. In the notation introduced in Section 2, the mathematical form of the model is (Rasch 1960, p. 171):

$$p(x_{ij}) = \frac{e^{(\theta_i+\sigma_j)x_{ij}}}{1 + e^{\theta_i+\sigma_j}}, \tag{5.18}$$

where the parameter, σ_j, describes item or question number j, and the parameter, θ_i, describes individual number i.

According to Theorem 1, the model (5.18) is uniquely identified by it's conditional probabilities,

$$p([x_{ij}] \mid r, c) = \frac{1}{\left[\begin{array}{c} r \\ c \end{array}\right]}, \tag{5.19}$$

where the denominator is the number of matrices $[X_{ij}]$ with marginals r and c. In the conditional distribution, (5.19) all outcomes have the same probability and it is therefore not possible to make an exact Rasch test as defined in Section 2. The same is the case for one of the generalizations of the dichotomous Rasch model that Rasch often worked with. But, as we shall see, for this generalization there is a way to make an exact Rasch test for selected parts of the data set. In this appendix, the model will be introduced and then the test will be derived. The derivations of the conditional probabilities are essentially given in an unpublished paper by Rasch and Stene (1967), but here, these derivations are placed within the framework of exact Rasch testing.

The model in question is aimed at analysis of questionnaires, where each single response category corresponds to a dimension or to a trait. We shall therefore assume that the locally independent random variables, X_{ij}, $i = 1, ..., I$, $j = 1, ..., J$ describes

individual number i's response to question/item number j in a questionnaire. Each item is assumed to have k response categories. In this paper we shall code the responses, X_{ij}, by k dimensional vectors, $X_{ij} = (X_{ij}^1, ..., X_{ij}^\mu, ..., X_{ij}^k)$ such that $X_{ij}^\mu = 1$ if person number i has chosen category μ when responding to item number j. Otherwise shall code X_{ij}^μ by zero. We therefore have that $X_{ij}^\mu \in \{0, 1\}$ and $X_{ij}^1 + \ldots + X_{ij}^k = 1$.

Since each category of an item represent a trait we have that each item, say item number j, is described by k unknown parameters, namely σ_{j1},...,σ_{jk}. We shall further allow the person's responses to the k traits to differ from trait to trait, that is, person number i is characterized by k unknown parameters, namely θ_{i1},..., θ_{ik}. The model which Rasch developed is:

$$p(X_{ij}^\mu = 1) = \frac{\theta_{i\mu}\sigma_{j\mu}}{\sum_{\mu=1}^k \theta_{i\mu}\sigma_{j\mu}} \tag{5.20}$$

As is the case with the dichotomous Rasch model (5.18), we get a unit distribution when conditioning on the row and column marginals (which in this case are matrices instead of vectors). It is therefore not possible to make an exact Rasch test by using these conditional probabilities.

We shall derive an exact Rasch test of model (5.20) for a selected part of the data set, namely for two items at a time and for individuals which have not responded identically to these two items. It will further be assumed that $k > 2$.

In this respect it may be noted that it is not a severe limitation that the test only applies to two items at a time. In practice, it is often a problem to find out which items fit the model and which does not. Especially so because an overall test of the model seldom gives information as to which specific items are the misfitting ones. Testing two items at a time does. Furthermore, in practice the number of items are often small, and it is therefore not too time consuming to perform the test for all combinations of pairs of items.

Now, let the two items under consideration be indexed by j_1 and j_2. Let the I persons be classified according to their responses to these two items, such that the random variable, Y_{gh}, denotes the number of persons who has responded in category g for item number j_1, and in category number h for item number j_2:

Let $X_{ij} = q_g$ be a short hand notation for $X_{ij}^g = 1$, that is, for cases where the respondent has chosen category number g. We shall start by modeling *one* persons

TABLE 5.1.

$j_2 \backslash j_1$	$x^1_{ij_1}$	$\cdots$	$x^g_{ij_1}$	$\cdots$	$x^h_{ij_1}$	$\cdots$	$x^k_{ij_1}$
$x^1_{ij_2}$	Y_{11}	$\cdots$	Y_{1g}	$\cdots$	Y_{1h}	$\cdots$	Y_{1k}
$\vdots$	$\vdots$	$\ddots$					$\vdots$
$x^g_{ij_2}$	Y_{g1}		Y_{gg}		Y_{gh}		Y_{gk}
$\vdots$	$\vdots$			$\ddots$			$\vdots$
$x^h_{ij_2}$	Y_{h1}		Y_{hg}		Y_{hh}		Y_{hk}
$\vdots$	$\vdots$					$\ddots$	$\vdots$
$x^k_{ij_2}$	Y_{k1}	$\cdots$	Y_{kg}	$\cdots$	Y_{kh}	$\cdots$	Y_{kk}

responses to the two items.

According to Theorem 1, the two locally independent random variables, X_{ij_1} and X_{ij_2}, are distributed according to (5.20) if and only if

$$P(X_{ij_1} = q_g, X_{ij_2} = q_h \mid X_{ij_1} + X_{ij_2} = q_g + q_h) = \frac{\sigma_{j_1 g}\sigma_{j_2 h}}{\sigma_{j_1 g}\sigma_{j_2 h} + \sigma_{j_1 h}\sigma_{j_2 g}}$$

$$= \frac{\frac{\sigma_{j_1 g}}{\sigma_{j_2 g}}}{\frac{\sigma_{j_1 g}}{\sigma_{j_2 g}} + \frac{\sigma_{j_1 h}}{\sigma_{j_2 h}}} = \frac{\delta_g}{\delta_g + \delta_h}$$

The case $h = g$ does not contain any information. Therefore we shall limit our attention to those individuals who did not respond identically to the two items, that is individuals, i, for which $X_{ij_1} = q_g$ and $X_{ij_2} = q_h$, $g \neq h$. Let $N_{gh} = Y_{gh} + Y_{hg}$. According to the principle of sufficiency the statistical model in which each individual is described by the probabilities $\frac{\delta_g}{\delta_g + \delta_h}$ is equivalent to the statistical model in which the distribution of (Y_{gh}, Y_{hg}) is a binomial, that is

$$P(Y_{gh} = y_{gh}, Y_{hg} = y_{hg} \mid N_{gh} = n_{gh}) =$$

$$\binom{n_{gh}}{y_{gh}} \left(\frac{\delta_g}{\delta_g + \delta_h}\right)^{y_{gh}} \left(1 - \frac{\delta_g}{\delta_g + \delta_h}\right)^{n_{gh} - y_{gh}} =$$

$$\binom{n_{gh}}{y_{gh}} \frac{\delta_g^{y_{gh}} \delta_h^{y_{hg}}}{(\delta_g + \delta_h)^{n_{gh}}}$$

Let $\langle Y_{gh} \rangle$ denote the set of random variables Y_{gh} for which $h \neq g$, that is, $\langle Y_{gh} \rangle$ is the random variables illustrated in Table 1 when the diagonal of the table is

discounted. Let $\langle N_{gh}\rangle$ denote the set of random variables N_{gh} for which $h \neq g$. Further, let Y_{g0} be the number of persons in $\langle Y_{gh}\rangle$, who have chosen category number g when responding to item number j_1, that is $Y_{g0} = \sum_{h\neq g} Y_{gh}$. According to the assumption of local independence combined with the arguments above we have that the persons in $\langle Y_{gh}\rangle$ are described by (5.20) if and only if

$$P(\langle Y_{gh}\rangle = \langle y_{gh}\rangle \mid \langle N_{gh}\rangle = \langle n_{gh}\rangle) = \prod_{g<h}\left(\binom{n_{gh}}{y_{gh}}\frac{\delta_g^{y_{gh}}\delta_h^{y_{hg}}}{(\delta_g+\delta_h)^{n_{gh}}}\right)$$

$$= \left(\prod_{g=1}^{k}\delta_g^{y_{g0}}\right)\prod_{g<h}\left(\binom{n_{gh}}{y_{gh}}\frac{1}{(\delta_g+\delta_h)^{n_{gh}}}\right)$$

According to Theorem 1 this is equivalent to

$$P(\langle Y_{gh}\rangle = \langle y_{gh}\rangle \mid \langle N_{gh}\rangle = \langle n_{gh}\rangle,\ (Y_{g0} = y_{g0})_{g=1,\ldots,k}) =$$

$$\frac{\prod_{g<h}\binom{n_{gh}}{y_{gh}}}{\Psi((y_{g0})_{g=1,\ldots,k} \mid \langle n_{gh}\rangle)} \tag{5.21}$$

where

$$\Psi((y_{g0})_{g=1,\ldots,k} \mid \langle n_{gh}\rangle) = \sum_{\{\langle \tilde{y}_{gh}\rangle \mid (\tilde{y}_{g0}=y_{g0})_{g=1,\ldots,k})\}}\ \prod_{g<h}\binom{n_{gh}}{\tilde{y}_{gh}}$$

In consequence an exact Rasch test for whether the persons in $\langle Y_{gh}\rangle$ can be described by the model (5.20) is obtained by using the conditional probabilities (5.21) as the test statistic and calculating the probability of significance as

$$\epsilon(\langle y_{gh}\rangle \mid \langle n_{gh}\rangle\ (y_{g0})_{g=1,\ldots,k}) = \sum_{C}\frac{\prod_{g<h}\binom{n_{gh}}{\tilde{y}_{gh}}}{\Psi((y_{g0})_{g=1,\ldots,k} \mid \langle n_{gh}\rangle)}$$

where $C = \{\langle \tilde{y_{gh}}\rangle \mid \langle \tilde{n_{gh}}\rangle = \langle n_{gh}\rangle, (\tilde{y}_{10},\ldots,\tilde{y_{k0}}) = (y_{10},\ldots,y_{k0}),$
$p(\langle \tilde{y_{gh}}\rangle \mid \langle \tilde{n_{gh}}\rangle,\ (\tilde{y}_{10},\ldots,\tilde{y_{k0}})) < p(\langle y_{gh}\rangle \mid \langle n_{gh}\rangle,\ (y_{10,\ldots,y_{k0}}))\}$.

7 Appendix B. Proof of Theorem 1

Theorem 1 has shown itself to be of great importance in this article. A proof of the theorem is available in Danish, but not in English, which is why a proof of Theorem

1 is included here. With minor exceptions the proof follows Gottschau (1983).

It is easily verified by straight forward computations that Equation (5.6) implies Equation (5.7) and Equation (5.8). It remains to be shown that Equation (5.8) implies Equation (5.6).

We shall consider a 2×2 sub-matrix of $[X_{ij}]$, namely:

$$X = \begin{pmatrix} X_{ij} & X_{i\rho} \\ X_{\tau j} & X_{\tau\rho} \end{pmatrix}$$

We shall consider two outcomes of X, namely

$$x^{A_t} = \begin{pmatrix} t-1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad x^{B_t} = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$$

where $t \in \mathbb{N}$ or $t \in \{1, ..., m\}$, depending on the sample space. The row and column marginals of x^{A_t} and x^{B_t} will be denoted respectively r^{A_t}, c^{A_t}, r^{B_t} and c^{B_t}. Only the case where $p(x^{A_t} \mid r^{A_t}, c^{A_t})$ and $p(x^{B_t} \mid r^{B_t}, c^{B_t})$ are strictly greater than zero is of interest.
We have that

$$\begin{aligned} & \frac{p(x^{A_t} \mid r^{A_t}, c^{A_t})}{p(x^{B_t} \mid r^{B_t}, c^{B_t})} \qquad (5.22) \\ = \; & \frac{p(x^{A_t})p(x^{A_t} \cap r^{A_t}, c^{A_t})p(r^{B_t}, c^{B_t})p(x^{B_t})}{p(x^{A_t})p(r^{A_t}, c^{A_t})p(x^{B_t} \cap r^{B_t}, c^{B_t})p(x^{B_t})} \\ = \; & \frac{p(x^{A_t})p(r^{A_t}, c^{A_t} \mid x^{A_t})p(r^{B_t}, c^{B_t})}{p(x^{B_t})p(r^{B_t}, c^{B_t} \mid x^{B_t})p(r^{A_t}, c^{A_t})}. \end{aligned}$$

Since x^{A_t} and x^{B_t} have the same row and column marginals we have that $p(r^{A_t}, c^{A_t}) = p(r^{B_t}, c^{B_t})$. Since further $p(r^{A_t}, c^{A_t} \mid x^{A_t}) = p(r^{B_t}, c^{B_t} \mid x^{B_t}) = 1$, the fraction (5.22) simplifies to $p(x^{A_t})/p(x^{B_t})$. Due to the assumption of conditional independence we have that

$$\frac{p(x^{A_t})}{p(x^{B_t})} = \frac{p(x_{ij} = t-1)p(x_{i\rho} = 1)p(x_{\tau j} = 1)p(x_{\tau\rho} = 0)}{p(x_{ij} = t)p(x_{i\rho} = 0)p(x_{\tau j} = 0)p(x_{\tau\rho} = 1)}$$

For fixed $\tau = \tau_0$ and $\rho = \rho_0$ we may denote

$$\theta_i = \frac{p(x_{i\rho_0} = 1)}{p(x_{i\rho_0} = 0)} \quad \text{and} \quad \sigma_j = \frac{p(x_{\tau_0 j} = 1)p(x_{\tau_0\rho_0} = 0)}{p(x_{\tau_0 j} = 0)p(x_{\tau_0,\rho_0} = 1)}$$

This implies that

$$\frac{p(x^{A_t})}{p(x^{B_t})} = \frac{p(x_{ij} = t-1)}{p(x_{ij} = t)}\theta_i\sigma_j.$$

which implies that

$$p(x_{ij} = t) = \frac{1}{\phi_{ii}(\theta_i, \sigma_j)}(\theta_i\sigma_j)^t\mu_{ij}(x^{A_t})$$

where

$$\mu_{ij}(x^{A_t}) = \prod_{\tilde{t}=0}^{t-1}\frac{p(x^{B_{\tilde{t}}})}{p(x^{A_{\tilde{t}}})}$$

and

$$\phi_{ij}(\theta_i, \sigma_j) = \sum_{t\in\Omega}(\theta_i\sigma_j)^t\mu_{i,j}(x^{A_t}).$$

8 References

Agresti, A., Wackerly, D. (1977), "Some Exact conditional tests of independence for $R \times C$ Cross-classification tables," *Psychometrica*, 42, 111-125.

Carr, W. E. (1980), "Fisher's Exact Test Extended to More Than Two Samples of Equal Size," *Technometrics*, 22, 269-270.

Gottschau, A. (1983), "En Analyse af et Objektivitetsbegreb. Med Udgangspunkt i Rasch's Psykometriske Arbejder," Copenhagen, Danmarks Pædagogiske Institut.

Radlow, R. and Alf, E. Jr. (1980), "An alternate multinomial assessment of the accuracy of the χ^2 test of goodness of fit," *Journal of the American Statistical Assciation,* 66, 755-759.

Rasch, G., Stene, J. (1967), "Some remarks concerning the inference about items with more than two categories". Unpublished.

Rasch, G. (2001), "On Objectivity and Specificity of the Probabilistic Basis for Testing," in *Rasch Lectures. In honor of Georg Rasch's 100 years birthday on the 21th of September, 2001,* ed. Olsen, L. and Kreiner, S., Copenhagen Business School,

2001.

Rasch, G. (1974), "On Specific Distributions for Testing of Hypotheses," in *Memoirs no.1 1974. Proceedings of conference on foundational questions in statistical inference,* Ed. O. Barndorff-Nielsen, P. Blæsild, G. Schou, University of Aarhus, Department of theoretical Statistics, pp. 101-110.

Rasch, G. (2001), "On Objectivity and Specificity of the Probabilistic Basis for Testing," in *Rasch Lectures. In honor of Georg Rasch's 100 years birthday on the 21th of September, 2001,* ed. Olsen, L. and Kreiner, S., Copenhagen Business School, 2001.

Appendix A. Pharmaceutical Prices in Europe: A Linear Structure

Tim Bedsted[1], **Kasper P. Jørgensen**[2], **Hans Keiding**[3] and **Lina Wøhlk Olsen**[4]

ABSTRACT In this paper we present a theoretical model as well as an empirical analysis of recent data for drug prices in 18 European countries. The data strongly suggest a linear price structure, under which the price of a drug has a component depending on its pharmaceutical properties and a country-specific component common for all types of drugs in this country. This linearity of the price structure reduces the significance of differences in the national price mechanisms and questions the impact of national interventions on drugs prices.

Using the linear structure we derive a pharmaceutical price index. Although the analysis is carried out on manufacturer-level drug prices, the countries in the study do not exhibit significantly different price levels at the pharmacy purchasing level.

1 Introduction

Over the past 25 years international price comparisons on pharmaceutical drugs has received attention from both policy makers as well as researchers. As mentioned by Danzon and Chao[4], such price comparisons have started from rather humble beginnings, using unweighed bundles of drugs, but have now evolved into price index computations using the Laspeyres or Paasche index formula. The focus in such comparisons has been on the differences of drug prices between countries and the resulting classification of countries as "expensive" or "cheap" from the point of view of the average consumer of pharmaceutical drugs. The structure of drugs prices within countries has received less attention.

In the present work, we approach the problem of comparing prices of pharmaceutical drugs from another angle, with the price structure playing an important role. This emphasis on price structure and its similarity in different countries is a consequence of a our findings based on price data for European countries collected in 1999: Except for random variation the prices of drugs in country A can be described

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as a fixed multiple of the prices in country B, so that the price in one country is a linear function of prices in other countries. This proportionality or linear structure is easily identified and consistent throughout the sample; we describe it in more detail in the sequel.

The findings are surprising in view of the fact that national pharmaceutical markets in Europe are highly regulated with regulations that differ considerably between countries. The linear structure of drug prices tells us however that the regulations influence only the level of prices but not the price structure. The proportionalities reproduce themselves at different levels of vertical structure, when the prices used are chosen as alternatively Pharmacy Purchasing Prices (PPP) as a reasonably good approximation of manufacturer-level prices or Pharmacy Selling Prices (PSP) at the retail level, the latter with and without VAT. The rules for assigning pharmacy mark-up to producer prices and the rates of either general or differentiated are vary considerably across the countries considered, but in the final picture they only add to the differences in levels without changing the structural relationship.

Given the linear price structure, one may construct a price index as the level factor of each country. The results of this price index computation are similar to those of other studies (e.g. Sermeus and Adriaessens[9]) in identifying a north-south division), but they broaden the perspective by isolating three different price groups. The largest group is a middle-price group, a somewhat smaller high-price group, and finally a small group, three countries, of low-price southern countries. Combined with the fact that we identify a more narrow distribution in price levels then earlier studies, we conclude that the overall price level in Europe is quite similar with a few anomalies such as Greece (being low) and Liechtenstein (being high). All in all the prices in Europe are quite homogeneous at the manufacturing level.

The paper is structured as follows: In Section 2, we present the data and the methods used in our analysis, and we present what may be seen as our most important finding, the linear structure of drugs prices. In the following Section 3, we discuss this linear price structure and give some theoretical arguments for its emergence. The following Section 4 describes the statistical analysis of the data, and in Section 5 we give a detailed presentation of the results of the empirical analysis. In Section 6, we discuss some methodological issues; finally, Section 7 contains a some concluding remarks.

2 Data and methods

Data for the pharmaceutical prices study reported in the present paper were collected with the purpose of computing indices of pharmaceutical drugs prices with Denmark as basis for the year 1999. The price information was collected by the Danish Medical Industry Assocation (LIF) as part of an agreement with the Danish Ministery of Health with the purpose of documenting the relation between drug prices in Denmark annd 17 other European countries. The European countries in the study are listed in table 6.1.

TABLE 6.1. The Countries in the Analysis

Austria	Greece	Luxembourg
Belgium	Holland	Norway
Denmark	Iceland	Portugal
Finland	Ireland	Spain
France	Italy	Sweden
Germany	Liechtenstein	UK

The information covering 18 countries specifies the 214 selected product packs in the following 6 dimensions: (1) brand, (2) molecule, (3) indication (ATC3 level) , (4) dosage form, (5) strength, and (6) pack size. Added to this is market information on (a) whether the brand product (item (1) above) was marketed in the 18 individual countries, (b) whether the product pack (defined at all 6 dimensions) was marketed in the various countries, and finally (b) the pharmacy purchasing price of the product pack in the 18 countries, valid at 01.03.99, which is the date of reference in the present study.

The matching of products was based on a tolerance of up to 50% on strength as well as pack size. In case of the match not being perfect a constructed price was calculated on the basis of the observed price. This was done by calculating a price per unit chemical substance and then correcting it to the Danish product pack size using the formula

$$\text{Constructed price} = \frac{s_h^{DK} u_h^{DK}}{s_h^J u_h^J} p_h^J$$

where p_h^J is the observed price of product h in country J , s_h^J is the amount of chemical substance per unit of product h in country J, and u_h^J is the pack size, number of units in each pack. Table 3 shows the distribution of perfect matches and constructed prices.

The additional Danish market information covers the following items for all 214

product packs: (1) Total sales in DDD, Moving Annual Total (MAT) and quarterly, (2) dispensation rule (3) reimbursement percentage, (4) year of introduction at the Danish market, (5) ATC classification (ATC5), and (6) whether or not there was generic competition.

The selection of product packs to be matched was based on the Danish market specifics. Three different procedures were used. The first procedure consisted in identifying the 100 largest generic product groups by January 1, 1996, ranked after MAT (Moving Annual Total) sales. The largest product pack in each group was then chosen, and the market information of these 100 product packs was then identified at the target date, March 1, 1999. Out of the original 100 product packs identified in 1996, 92 was refound in 1999. In the second procedure the 100 largest generic groups were identified at the target date March 1, 1999, and the third and final procedure consisted in choosing the 125 largest individual product packs at the same target date.

Pooling these three samples added up to 214 product packs. For 164 it was possible to find one or more counterparts in other countries. The total sample amounts to 40% of total market measured in DKK MAT sales, but only 5% of total quantities measured in DDD (Defined Daily Dosis) MAT sales.

In addition to prices, the data provide us with characterizing information about (a) sales, (b) dispensation rule, (c) reimbursement level, (d) introduction year, (e) ATC classification, and (d) indication of generic competition. Moreover, it is possible to construct the following variables: (i) share of constructed prices for each country, (ii) penetration of product (number of countries in which the product is found). These variables will be used as *exogenous variables* in the statistical analysis of the model.

3 A Linear Structure of Pharmaceutical Drugs Prices

One of the striking features of the data is revealed by a graphic analysis, where prices of the matched products from two different countries are plotted against each other. This is done for all possible pairs of countries, giving a total of $18(18-1)/2 = 153$ plots. All plots support the linear structure between all 18 countries. One of these plots, for Germany versus Portugal, is shown in Figure 1; the full collection of plots can be found at the website `www.pubhealth.ku.dk/~hake/drugsprices`.

The structure emerging from Figure 1 (and the remaining 152 plots) may be

TABLE 6.2. Overview of Product Match and Constructed Prices

Country	Direct Match	Constructed Price	Total Match	Share of Sample (164)	Constructed Prices Share of Match
Austria	71	28	99	0,60	0,28
Belgium	56	35	91	0,55	0,38
Finland	121	13	134	0,82	0,10
France	42	29	71	0,43	0,41
Germany	98	19	117	0,71	0,16
Greece	28	13	41	0,25	0,32
Holland	77	32	109	0,66	0,29
Iceland	109	9	118	0,72	0,08
Ireland	65	39	104	0,63	0,38
Italy	22	21	43	0,26	0,49
Liechtenstein	81	14	95	0,58	0,15
Luxembourg	50	20	70	0,43	0,29
Norway	108	17	125	0,76	0,14
Portugal	51	27	78	0,48	0,35
Spain	52	33	85	0,52	0,39
Sweden	107	19	126	0,77	0,15
UK	73	31	104	0,63	0,30

expressed as

$$P_h^B = \delta_A^B P_h^A, \tag{2}$$

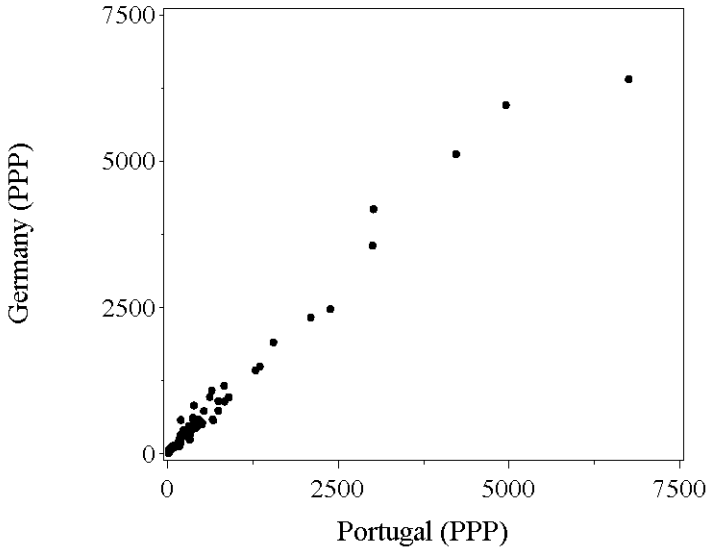
where P_h^J is the price (observed or constructed according to (1) above) of drug h in country J, and A and B are two countries. The factor of proportionality δ_A^B (which is the slope of the stylized straight line in the plot) then indicates which country is the more expensive; the main point is that δ_A^B depends on the countries A and B, but *not* on the drug h. Choosing prices in country A as reference and writing $\delta_A^B = \delta_B$ and $P_h^A = \theta_h$, the equation (2) may alternatively be written as

$$P_h^B = \delta^B \theta_h, \tag{3}$$

a formulation which stresses the multiplicative (or linear) structure of drugs prices, that is the price of any drug in any country may be obtained as a product of a component which is country-specific and another one which is drug-specific.

From Figure 1 it is clear that the price equation in (3) is only valid in average, so we reformulate it to obtain a statistical model which takes the random variations into account. Let $X_h^B = \log P_h^B$ denote a random variable that describes the logarithm

FIGURE 1. Plot of drugs prices in two countries: Germany versus Portugal



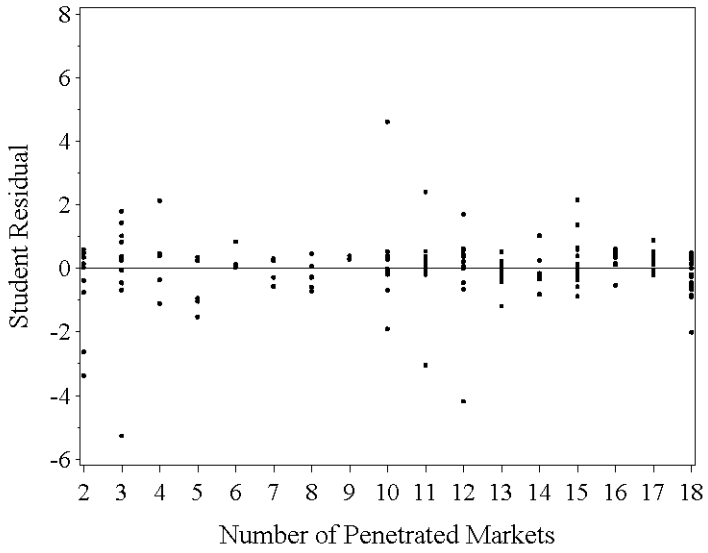
of the price on drug h in country B. Then our probabilistic version of (3) may be written as

$$X_h^B = \beta^B + \alpha_h + u_h^B, \tag{4}$$

where u_h^B is a disturbance term, and α_h and β^B are the parameters of the statistical model. We shall assume that the error terms are independent.

The relationship in (4) applies to all drugs. However, there are certain cases which are particular: It can be verified that there is an almost complete match between Luxembourg's and Belgium's prices. This can be explained by the simple fact that Luxembourg has a price control rule, which sets the maximum price of the individual products as that of the corresponding Belgian, cf. Mossialos and Le Grand[8]. Including both countries in the statistical model would produce a violation of the independence assumption, so we have removed the country Luxembourg from the statistical analysis.

FIGURE 2. Residual plot for the market penetration variable



Concerning the issue of missing values due to non-match of products, it seems to be a reasonable assumption that whether or not an observation X_h^B is missing does not depend on the value of the observation, but rather on which specific country C^B and which specific drug D_h the observations relate to. We therefore assume that missing data are missing at random. Under this assumption the parameters may be estimated using the data available (Little and Rubin[6]).

It seems reasonable in view of the graphical analysis to assume that the mean value of the disturbance term equals zero for all products and all countries B. Therefore, we may obtain estimates of α_h and β^B using least squares. However, in order to construct confidence limits, and test for whether the difference from the Danish price structure is significant, we need further assumptions about the distribution of the error term. A common assumption is that the error terms are independent and normally distributed with mean zero and variance σ^2 (identical for all B and h). Under these assumptions the model in (4) can be identified with that of two-sided

analysis of variance. The SAS procedure 'proc glm' may be used to calculate the maximum likelihood estimates, also in cases of unbalanced designs, which, due to the missing values, is the case for our data, and the likelihood estimates coincide with the the ordinary least square estimates. The estimates of the country parameters are displayed in Tables 6.3, 6.4, and 6.5, where the parameters for Denmark are fixed to zero.

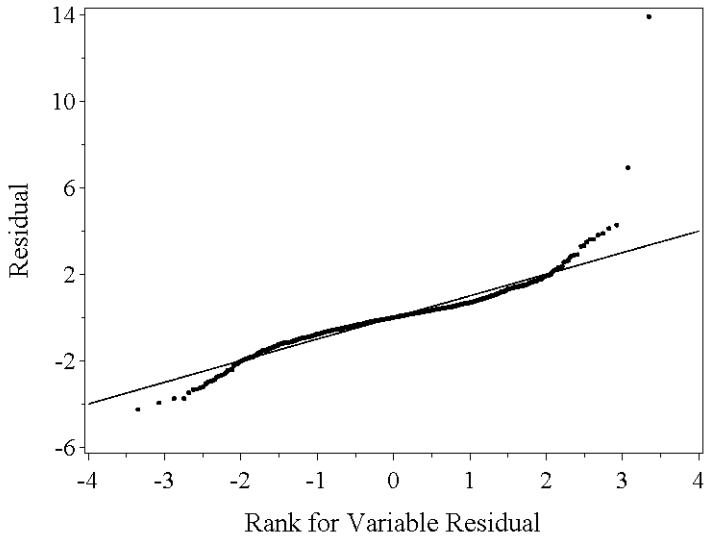
The R-square is 0.976, showing that the amount of variance explained by the model is high. The model assumptions, namely that the error terms are independent and identically normally distributed with mean zero and homogenous variance, must be checked before we can judge how well our model fits the data. However, the assumptions made are not equally important. The assumption about the normal distribution is only necessary when constructing confidence limits and testing differences between countries. In contrast, the assumption about zero mean is not only necessary to obtain central estimates; it questions the fundamental price equation (4).

One of the plots of the Student residuals against exogenous variables, in this case the variable (f) market penetration measured as the number of markets where the product is found, is shown in Figure 2 (the remaining plots can be found at `www.pubhealth.ku.dk/~hake/drugsprices`). The Student residuals are estimates of the error terms, standardized to unit variance, and therefore they should scatter unsystematically around the x-axis, with approximately 95 percent of the observations lying inside the boundaries $[-1.96, 1.96]$, and 5 percent outside. It may be checked that the residual plots show no clear deviations from this pattern, so that the assumptions of mean zero and homogenous variance of the error terms may be upheld, and we have found no significant deviations from the price equation.

The assumption about normally distributed error terms may checked by means of a QQ plot, cf. Jobson[5]. If the error terms are normally distributed with mean zero and unit variance the plot will result in a straight line with slope one, intersecting the ordinate in zero.

As is seen from Figure 3 this is not quite the case. More specifically, the tails of the distribution of the error terms are a little too heavy to match a normal distribution. In this respect it may be noted that the relatively large number of students residuals, namely 1703, makes it possible to identify even small derivation from the normal. All in all, the derivations are considered to be relative small compared to the large number of observations, and the normal distribution is regarded adequate to account for the variation of the error terms. However, since the normal distribution clearly

FIGURE 3. QQ plot for checking normality



does not describe the variation perfectly, the precision of the confidence limits and tests of difference amongst countries must not be overstated.

4 Indices of pharmaceutical drugs prices

The estimates of the model shown in (4) can further be transformed into a price index. We have chosen Denmark as our point of reference, meaning that Denmark is fixed to 1.00. The standard errors amy be used to derive confidence limits, which are also listed in Table 6.3.

Looking at Table 6.3 we find three distinct price level groupings, namely a low price level group with 4 countries (Greece, Italy, Spain and Portugal), a larger group of 8 countries with middle range price levels, comprised of several Central European, Northern and Scandinavian countries, also including France, and finally, a group of 6 high price level countries with some small countries (Iceland and Liechtenstein) as

well as large ones (Germany and UK). It is seen that the spread in prices is relatively high, from 0.68 to 1.30.

TABLE 6.3. Index of pharmaceutical drugs prices. Pharmacy purchasing prices (PPP), Denmark = 100.

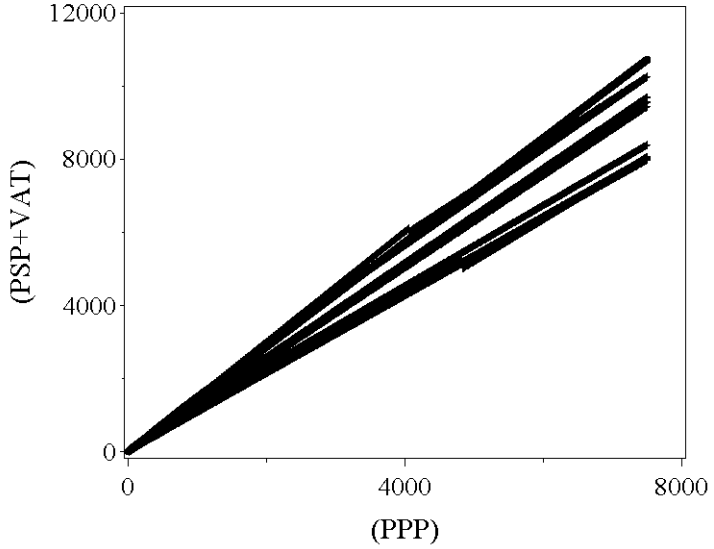
Rank	Country	Estimate	Std.	p-value	Index	Confidence Limits
1	**Greece**	-0.38898	0.037564	< 0.0001	0.67775	[0.72953, 0.62964]
2	**Spain**	-0.18634	0.028733	<0.0001	0.82999	[0.78454, 0.87808]
3 (1)	Portugal	-0.13653	0.029481	<0.0001	0.87238	[0.82340, 0.92427]
4 (2)	Italy	-0.10184	0.036856	<0.0058	0.90318	[0.84023, 0.97083]
5 (3)	Finland	-0.02124	0.024604	0.3880	0.97898	[0.93289, 1.02735]
6 (4)	Belgium	-0.00863	0.027967	0.7576	0.99140	[0.93852, 1.04726]
7 (5)	Luxembourg**	.	.	.	0.99140	[., .]
8 (6)	Denmark	0.00000	.	.	1.00000	. .
9	**France**	0.00194	0.030471	0.9494	1.00194	[0.94385, 1.06360]
10 (7)	Sweden	0.01711	0.025098	0.4955	1.01726	[0.96843, 1.06855]
11 (8)	Norway	0.03353	0.025132	0.1824	1.03410	[0.98439, 1.08631]
12 (9)	Austria	0.04884	0.027201	0.0727	1.05006	[0.99554, 1.10756]
13 (10)	Holland	0.07758	0.026377	0.0033	1.08067	[1.02622, 1.13801]
14	**Ireland**	0.11304	0.026765	<0.0001	1.11967	[1.06245, 1.17998]
15 (11)	Germany	0.15420	0.025790	<0.0001	1.16672	[1.10921, 1.22721]
16 (12)	Iceland	0.16626	0.025619	<0.0001	1.18088	[1.12304, 1.24169]
17	**UK**	0.20240	0.026776	<0.0001	1.22434	[1.16174, 1.29031]
18	**Liechtenstein**	0.25963	0.027556	<0.0001	1.29645	[1.22829, 1.36840]

We have excluded Luxembourg from the statistical analysis as discussed in Appendix **??, however this was done since the data were identical with the Belgium data, which on the other hand exactly points to the fact that the price levels of the two countries are identical.

The identified linear structure at the PPP level is kept in the downstream price levels of PSP with and without VAT. This finding is based on information on the respective national pharmacy mark-up models, and the respective national VAT on pharmaceutical drugs; thus, the PSP with and without VAT used here is a calculated price based on the observed PPP. Having the calculation models of the PSP with and without VAT enable us to analyze the structural as well as the level effects directly. An overview of the various national calculation models can also be found at `www.pubhealth.ku.dk/~hake/drugsprices`.

Despite large differences in the various versions of national pharmacy mark-up models they are all at least approximately affine functions. The PSPs are in other words linear transformations of the PPPs and do therefore not change the linear price

FIGURE 4. Pharmacy Mark-up Models



structure identified at the PPP level. From results below, on the change in rankings between the three vertical prices, we will see, however, that they have significant level effects. To illustrate these two points we have calculated the mark-up models for the 11 countries in question on the full price range covered in the sample. This is done in Figure 4. From Figure 4 we see that there is a clear dispersion of pharmacy mark-up levels, and, more importantly, that 9 of the 11 models are clearly linear whereas two are approximately linear.

The level and use of VAT follows the picture of the mark-up models inasmuch as the different countries display systems with huge differences covering the range from PPPs exempted VAT to 25% VAT on the PPPs. However, it is clear that the VAT will only have level effects and thereby support the identified structure.

In order to ease the comparison of the various price levels we have replicated the calculations leading to the results in Table 6.3 for the two prices of PSP with and without VAT; the prices could also have been calculated directly by applying the

various affine transformations. The results are presented in Table 6.4 and Table 6.5. The following discussion on the relative prices will consequently be based on a comparison of the the results in Tables 6.3, 6.4 and 6.5.

Information on PSP with and without VAT was not found for France, Greece, Ireland, Liechtenstein, Luxembourg, Spain, and the UK. These countries are therefore excluded from the following discussion. In order to ease the comparison of the three tables the countries excluded in Table 6.3 are typed in boldface.

Using the results in Table 6.3 as a point of departure the addition of pharmacy mark-ups has the overall effect of increasing the span in price levels from 0.87 – 1.18 to 0.84 – 1.33. The different national pharmacy mark-up systems therefore seem to support the dispersion of prices on pharmaceutical products in Europe. Furthermore this effect is not homogeneous and has clear implications on the resulting country ranking. The national choice of VAT does not alter the overall picture with respect to price level diversion, but has implications on the resulting ranking.

In both Table 6.4 and in Table 6.5 it is seen that the split between countries with a price level significantly higher and significantly lower than the Danish one remains unchanged, although some countries move between the groups of low, middle, and high price level. Except for the two low price level countries Portugal and Italy and the two high price level countries Germany and Iceland, which all consistently stay in the extreme ends, there are significant changes in the ranking of the various countries when adding the pharmacy mark-ups and VAT.

TABLE 6.4. Index of pharmaceutical drugs prices. Pharmacy sales prices (PPP). Denmark = 100.

Rank	Country	Est.	Std.	p-value	Index	Confidence Limits
(1)	Portugal	-0.16952	0.027286	<0.0001	0.84407	[0.80011, 0.89044]
(2)	Italy	-0.13005	0.034245	0.0002	0.87805	[0.82105, 0.93901]
(3)	Norway	-0.06051	0.023186	0.0092	0.94129	[0.89947, 0.98505]
(4)	Belgium	-0.04459	0.025856	0.0849	0.95639	[0.90913, 1.00611]
(5)	Sweden	-0.01220	0.023156	0.5983	0.98787	[0.94404, 1.00611]
(6)	Austria	-0.00113	0.025125	0.9641	0.99887	[0.95087, 1.04929]
(7)	Denmark	0.00000			1.00000	
(8)	Holland	0.02323	0.024341	0.3400	1.02351	[0.97582, 1.07352]
(9)	Finland	0.13989	0.022684	<0.0001	1.15015	[1.10014, 1.20244]
(10)	Germany	0.22780	0.023781	<0.0001	1.25583	[1.19864, 1.31575]
(11)	Iceland	0.28408	0.023629	<0.0001	1.32854	[1.26842, 1.39152]

All in all the succeeding vertical price levels to PPPs illustrate very large differ-

TABLE 6.5. Index of pharmaceutical drugs prices. Pharmacy sales prices, VAT included. Denmark = 100.

Rank	Country	Est.	Std.	p-value	Index	Confidence Limits
(1)	Italy	-0.35320	0.034245	<0.0001	0.70244	[0.65684, 0.75120]
(2)	Portugal	-0.34388	0.027286	<0.0001	0.70902	[0.67209, 0.74797]
(3)	Sweden	-0.23535	0.023156	<0.0001	0.79030	[0.75523, 0.82699]
(4)	Belgium	-0.20946	0.025856	<0.0001	0.81102	[0.77094, 0.85318]
(5)	Holland	-0.14164	0.024341	<0.0001	0.86793	[0.82750, 0.91035]
(6)	Norway	-0.07663	0.023186	0.0010	0.92623	[0.88508, 0.96929]
(7)	Austria	-0.01726	0.025125	0.4922	0.98289	[0.93566, 1.03250]
(8)	Denmark	0.00000			1.00000	
(9)	Finland	0.03008	0.022684	0.1851	1.03054	[0.98572, 1.07739]
(10)	Iceland	0.06094	0.023629	0.0100	1.06283	[1.01473, 1.11322]
(11)	Germany	0.15307	0.023781	<0.0001	1.16541	[1.11234, 1.22102,]

ences between national price systems but also that these differences do not have any structural effects. Only level effects can be observed, these being quite dramatic with respect to the change in rankings dependent on the use of vertical prices.

5 Theories of oligopolistic pricing and the linear price structure

The empirical findings discussed in the previous sections show that relative drugs prices are largely the same in the countries of Europe while the price level may differ considerably. This may be surprising at a first glance; according to the theory of oligopolistic pricing, we do not expect prices to be identical to what they would have been in perfectly competitive markets, in which case one would have an approximate world market price in each country, something not very far from what we found. But drugs markets are very far from being competitive, and the possibility of arbitrage which would result in identical relative prices is only available to a very limited extent (the system of parallel imports practiced in some European countries). In oligopolistic markets we would expect prices to reflect the situation in each national market (involving both the behaviour of the competitors and the demand of the consumers, which in our case is reflected in differences in prescription practice). The fact that none of this shows up in the data needs an additional reflection.

Intuitively, it seems plausible to assume that the common price structure is a result of a common price decision, so that the drugs producers set prices (relative to the prices of other drugs) for Europe as a whole rather than for each single country separately. On the other hand, it still remains to be explained why they do so; it

would seem to be rational for each firm to depart from a common rule when the national market characteristics are favourable to such a departure, either due to higher consumer interest in the product or to less active presence of competitors.

A possible answer to this puzzle is furnished by the theory of conjectural variations using the repeated games approach (cf. e.g. Tirole [11]): Assume that market demand for a good is given by

$$p_J = a_J - b_J q_J,$$

where p_j and q_J are price and quantity demand for the good in country J, and $a_J, b_J > 0$ are constants. If there are several, say k firms selling the good, all producing at unit costs c, then joint profit is maximized if the firms all charge the monopoly price given by

$$p_J^m = \frac{a_J + c}{2}$$

and together sell $a_J - c/2b_J$, which gives each of the firms a profit of $\Pi_J^m = (a_J - c)^2/2kb_J$ (assuming an equal split of the customers). Clearly, this arrangement is highly unstable; by charging a price slightly lower than p_J^m any firm can attract all the customers from the remaining firms and thereby obtain (almost) k times the profit obtained. The only stable situation is the (Bertrand) equilibrium where every firm sells at $p_J = c$ (the outcome obtained under perfect competition).

What may deter a firm from undercutting price is the conjectured response of the other firms. Indeed, suppose that the firm expects the competitors to cut prices all the way down to c in the future as a reaction on any initial price cut. Then the immediate gain should exceed the future losses at the discount rate δ:

$$\frac{k-1}{k}\,\Pi_J^m > \sum_{t=1}^{\infty} \delta^t \frac{1}{k}\,\Pi_J^m = \frac{\delta}{1-\delta}\frac{1}{k}\,\Pi_J^m$$

for a price cut to be advantageous, and this will happen only if the discount rate satisfies $\delta < (k-1)/k$.

Actually, the firm may rationally expect a less drastic response than the one discussed above (the so-called "trigger strategy" of the repeated game literature), and in that case the temptation to engage in price cuts may be a real one; however, in our case, where the firms are selling the same products in several markets (and facing the same competitors in all these markets), the retaliation by the others to a price cut in any single country J would probably be price cuts (even if temporary) in *all* markets rather than just in market J. In the case where all play the trigger

strategy we get the inequality

$$\frac{k-1}{k}\,\Pi_J^m > N\sum_{t=1}^{\infty}\delta^t\frac{1}{k}\,\Pi_J^m,$$

or equivalently

$$\delta < \frac{k-1}{k+N-1}$$

to be satisfied for the price cut to be advantageous. It should be noticed also that if a price cut in country J is advantageous then it is a fortiori advantageous to engage in price competition in all markets at the same time, since anyway retaliation by the others will involve all these markets.

While this theory provides an argument for the existence of a stable pricing equilibrium different from the competitive level, it does not explain how this stable price is chosen; indeed, the stability due to fear of retaliation would apply to any level of prices (this is an instance the celebrated Folk Theorem for repeated games (cf. e.g. Sorin[10]). In this situation, one usually falls back on what is called a *focal* price as the explanation of the actual price level: There is some specific price (such as the joint profit maximizing monopoly price) which comes to the mind of the competitors more readily than the others and which therefore is the one to which everyone sticks. Even if this may not in itself a particularly convincing explanation, it does make sense to argue that in our case, where the markets are so intimately connected, a focal price must be common for all countries except for a market specific exchange rate, and this will indeed yield the linear structure which turns up in the data.

Some suggestions as to what may constitute a focal price suggest themselves: When a drug is first introduced, the level of competitiveness is usually low, and the optimal price may come close to the monopoly level; when other producers eventually enter the market, the mechanism above applies to keep the price at the original level; indeed, all the competitors will be reluctant to cut prices so that it will remain at the originally fixed level. This type of stickiness seems to fit rather well with the experience from national drug markets. Also, it should be mentioned that regulation based on European averages is very commonplace in the countries considered; such regulation will have the effect of making the producers increasingly aware of interconnection between individual European markets, eventually treating them as a single market, with the uniformity of price structure as a natural consequence.

TABLE 6.6. Penetration of national markets for the products of the sample

Number of national markets	Frequency (Number of products)	Percentage in sample	Cumulative frequency	Cumulative percentage
1 (DK)	50	23.36	50	23.36
2	9	4.21	59	27,57
3	15	7.01	74	34,58
4	7	3.27	81	37,85
5	6	2.80	87	40,65
6	4	1.87	91	42,52
7	5	2.34	96	44,86
8	6	2.80	102	47,66
9	4	1.87	106	49,53
10	14	6.54	120	56,07
11	11	5.14	131	61,21
12	11	5.14	142	66,36
13	15	7.01	157	73,36
14	7	3.27	164	76,64
15	17	7.94	181	84,58
16	7	3.27	188	87,85
17	11	5.14	199	92,99
18	15	7.01	214	100,00

6 Discussion

The findings reported in the previous sections, in particular the linear structure of drugs prices in European countries, are such that particular care must be taken to assure that they are not caused by peculiarities of the sample selection, which might reduce its representativeness. In this section, we consider the sample, already mentioned in Section 2, somewhat more closely.

Table 6.6 shows the overall penetration of national markets of the products in the sample. Here we find that almost a quarter of the total sample is marketed only in Denmark. Of the remaining 164 product 1/3 is marketed in half or less countries and only 7% of the total sample is marketed in all 18 countries. The sample is highly differentiated in the degree of international diffusion. With approximately a quarter of the sample being strictly Danish products and the remaining group being a mix of regional and international products ranging over the full possible spectrum.

Our data contain the introduction time for each product in Denmark, as shown in table 6.7. The variation is quite high; approximately one quarter of the sample

TABLE 6.7. Distribution of Product Age in Countries

Row Pct.	–1975	1975–84	1985–89	1990–94	1995–	Total
	Pct.	Pct.	Pct.	Pct.	Pct.	Freq.
Austria	1,01	4,04	25,25	40,40	29,29	99
Belgium	2,20	7,69	26,37	39,56	24,18	91
Denmark	5,49	10,98	24,39	39,63	19,51	164
Finland	4,48	9,70	25,37	41,04	19,40	134
France	1,41	7,04	25,35	38,03	28,17	71
Germany	1,71	6,84	23,93	41,88	25,64	117
Greece	4,88	4,88	19,51	41,46	29,27	41
Holland	1,83	5,50	22,02	44,04	26,61	109
Iceland	4,24	11,86	22,88	39,83	21,19	118
Ireland	2,88	8,65	24,04	41,35	23,08	104
Italy	2,33	9,30	23,26	30,23	34,88	43
Liechtenstein	2,11	5,26	21,05	45,26	26,32	95
Luxembourg	2,86	11,43	24,29	37,14	24,29	70
Norway	5,60	11,20	23,20	36,80	23,20	125
Portugal	2,56	6,41	21,79	41,03	28,21	78
Spain	2,35	7,06	25,88	38,82	25,88	85
Sweden	3,17	13,49	23,02	38,89	21,43	126
UK	2,88	7,69	22,12	40,38	26,92	104
No-match	20,00	4,00	18,00	32,00	26,00	50

is younger than five years whereas the largest group those between 5-10 years old all lie in the region of 35% whereas the group of products between 10 and 15 years covers close to a quarter of the respective samples.

The most obvious difference is between the no-match sample and the remaining samples. However this seems to be particularly explicit in the group of products older than 10 years where the no-match group a significantly higher share of the more than 25 year old products.

Summarizing on the characteristic of the 214 products in our sample, it seems that the products are slightly younger than the Danish domestic products and when turning to the difference to the overall market the sample seems to consist of products which are significantly more expensive on average than the whole market average. This was seen also in section 2 where it was mentioned that the sample cover 40% of total sales but only 5% of total quantities.

The stylized fact of a north-south division with regard to European pharmaceutical drug prices, cf. e.g. Andersson[1], may perhaps be extended to therapeutic differences. We have several indications of this, among which in particular the number of

matches/non-matches indicating differences in product brand demand and preferences regarding the packages. This is confirmed by table (6.2) showing a geographical dependent match of brand products, the north-south division. This systematic behavior is less clear when looking at the share of corrected prices for each country. Here the Nordic countries (Denmark, Norway, Sweden, Finland and Iceland) clearly differs from the southern European countries but countries inbetween (i.e. Germany and Holland) seems to fall randomly into either of the two groups.

A potential flaw of our analysis and the identification of a linear structure in prices may arise if the results are based on stereotyped samples. The literature on pharmaceutical price comparisons has dealt with the cases of biased sampling. One may identify two main issues in this context: The first one is the impact of national differences in regulatory systems, competition setup, and therapeutic tradition and is presented e.g. in Andersson[2]. The second one, analyzed in Danzon and Kim[3], Danzon and Chao[4], is concerned with mechanisms causing structural differences: different price regulation in countries may lead to different presence in the sample of generic products.

According to Andersson[2], to be able to perform price comparisons which are not skewed through structural differences, the underlying price structure must not be too different. Reversing the argument, the identical price structure identified in this paper might be caused by the fact that the compared countries are not structurally different in the relevant dimensions. This argument is met by Mossialos and Le Grand[8]. Mossialos and Le Grand show that the countries in this study differ in all possible dimensions regarding regulatory systems and competition setup, and they also indicate fundamental differences in therapeutic tradition. Therefore we may assume that our findings are not a result of identity in the national institutional setups.

The issues raised in Danzon and Kim[3], Danzon and Chao[4] may be treated much in line with the one raised above, inasmuch as it draws on the differences in national price regulation. Danzon e.a. show that the prices of old molecules, and prices of generic products, are highly asymmetric between countries and follow the general level of price regulation. Countries with strong price regulation have systematically higher prices on old products as well as on generic products.

Looking first at the distribution of product age in our sample, which is presented in Table 6.7, it may be seen that all the age groups are represented in each of the countries of the study. The sample is thereby not stereotyped in this respect, but spans the full outcome space described in Table 6.7.

We may take this a little further. Using the statistical version of the identified linear structure we may detect any systematic deviation of observations from the expected value of the model. Plots of the standardized residuals against any exogenous variable will display systematic behavior in this variable and thus work as a general model test. One of these residual plots can be found in Figure 5. For each country we have residual plots of the age variable, and it may be seen that there is no systematic behavior (the full collection of plots can be found at the website: `www.pubhealth.ku.dk/~hake/drugsprices`). Since we have a broad variety of product ages in our sample and the residuals display no systematic behavior, we conclude that our sampling is not biased with respect to product age.

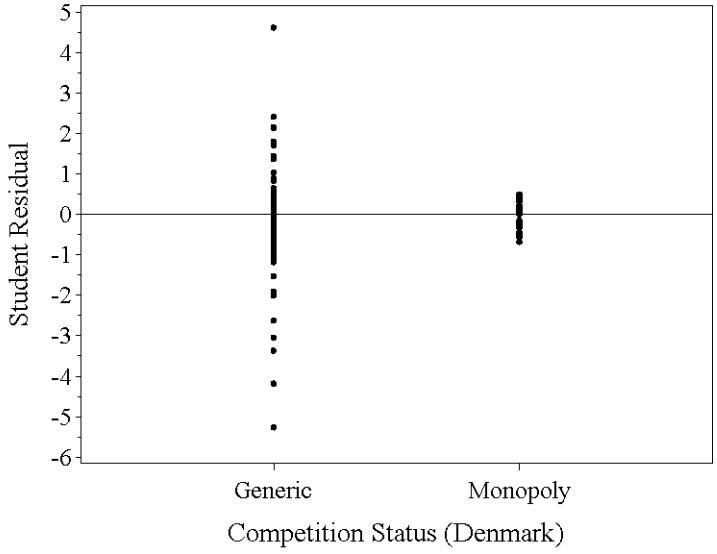
Turning to the question of whether markets for generic products are sufficiently well represented in the sample, we note that our information about generic products is derived from the Danish market. Testing the findings in Danzon et al. is therefore not as straightforward as in the case above, since the generic variable is not as global as the age variable. However, it seems reasonable to assume that since Denmark has a highly developed generic market, cf. e.g. Mossialos & Le Grand[8], any monopoly market in Denmark is also a monopoly market in the other European countries. In the residual plot covering the monopoly variable, cf. Figure 5 for the countries, we do not find any systematic behavior in the generic, and more importantly, the monopoly variable, and conclude that the linear price structure found in this study is not a result of biases in the generic mechanism.

Looking finally at the remaining exogenous variables (market penetration, product match/constructed prices, dispensation (in DK), reimbursement level (in DK), and ATC classification) it is clear that these variables vary in degree of how global they are. Information from the Danish market on product age and ATC-classification are more global indicators then whether the products are OTC (Over The Counter) or POM (Prescription Only Market) products in Denmark. Having this in mind we find nonetheless that the respective country samples span the full outcome space of the respective variables[5] and furthermore that none of the variables display any systematic behavior in the relevant residual plots.

We conclude that as far as we may investigate the sample is not biased in the available exogenous variables, and furthermore that none of these indicate flaws in the presented model. All in all it is not possible to discharge the linear finding on

[5]In the case of the ATC classification this is true with minor exceptions for the low match countries in the very small ATC-groups.

FIGURE 5. Residual plot for competitive status



the grounds of stereotyped products.

7 Concluding remarks

In the present paper, we have investigated the structure of pharmaceutical drugs prices based on a sample of drugs selected from the Danish pharmaceutical market, supplemented with price information from 17 European countries. It was found that the pharmaceutical prices display a *linear structure* in the sense that the price of a given drug in a given country can be found approximately as the product of a product factor and a country factor.

The implications of these findings may be rather far-reaching: If the price structure has this form, then the effects of the very different types of price regulation implemented in different European countries seem to have had a very limited impact on the actual prices, at most influencing the country factor but not the relative

prices. It may be added that an overall effect of regulation on the national price level, without changes in relative prices, seem highly implausible for many of the regulations actually carried out.

Turning to the vertical price structure (pharmacy sales prices with and without VAT), it turns out that the very different rules for pharmacy profit mark-ups and VAT change the ranking of countries with respect to average level, but do not change the relative prices. Thus the linear structure detected for pharmacy purchasing prices is reestablished at the downstream levels. The changes in ranking show that the choice of vertical level matters when performing international comparisons of drugs prices.

8 References

[1] Andersson F, McMenamin P. International Price Comparisons of Pharmaceuticals - A Review of Methodological Issues. Report. Battle Medical Technology and Policy Research Centre, 1992.

[2] Andersson F. Methodological Aspects of International Drug Price Comparisons. Pharmacoeconomics 1993; 4 (4): 159 – 195.

[3] Danzon PM, Kim JD. International Price Comparisons for Pharmaceuticals: Measurement and Political Issues. Pharmacoeconomics 1998; 14 (S1): 115 – 128.

[4] Danzon PM, Chao L. Cross-national price difference for pharmaceuticals: how large and why. Journal of Health Economics 2000; 19: .

[5] Jobson JD. Applied Multivariate Data Analysis, Volume I: Regression and Experimental Design. New York, Springer-Verlag, 1991.

[6] Little RJA, Rubin DB. Statistical analysis with missing data. New York, Wiley, 1987.

[7] LIF (Danish Pharmaceutical Association): 'Tal og Data'. Report from the Danish Pharmaceutical Association. 2001.

[8] Mossialos E, Le Grand J. Health Care and Cost Containment in the European Union. England, Ashgate Publishing Company, 1999.

[9] Sermeus G, Adriaessens G. International drug price comparison. In Drug prices and drug legislation in Europe - an analysis of the situation in the twelve mem-

ber states of the European Communities, Report from Belgian Consumers Association (BEUC), Brussels. 1989.

[10] Sorin S. Repeated games with complete information, chapter 4 in: Aumann J, Hart S (eds). Handbook of Game Theory, Elsevier Science Publishers. 1992.

[11] Tirole J. Theory of Industrial Organization, Cambridge, MIT Press, 1989.

Appendix B. List of Rasch's publications and important papers.

It seems to be a well established fact that Rasch was not that keen on finishing his contributions for publications. Some of his his discoveries were, however, circulated as duplicated papers and from time to time he would get other scientists to finish his papers for him. Furthermore, when working as a statistical consultant it was often the case that Rasch's statistical analysis was published as a (separate) part of the clients doctoral thesis or article, and this way Rasch actually published quite a lot, though he was not always contributed as author. In consequence, a list of Rasch's publications will not quite illustrate his scientific contributions. On the occasion of Rasch's 70 years birthday a list of his most important papers and contributions was made.[1] This list has in this Appendix been combined with a similar list in Rasch (1980).

1. Niels Nielsen in cooperation with G.Rasch (1923). Notes Supplémentaires sur les Equations de Lagrange. *Det Kgl. Danske Videnskabernes Selskab. Matematisk-fysiske Meddelelser,* V, 7, pp. 24.
2. G. Rasch (1927). Beitrag zur Theorie der unvollständigen Gammafunktionen. *Det Kgl. Danske Videnskabernes Selskab. Matematisk-fysiske Meddelelser.* VIII, 2. pp. 20.
3. G. Rasch (1928). Über die Nullstellen der unvillständigen Gammafunktionen $P(z, \rho)$.I. *Matematische Zeitschrift,* 29, p. 300-318.
4. E. Hille und G. Rasch (1928). Über die Nullstellen der unvillständigen Gammafunktionen $P(z, \rho)$. II. *Matematische Zeitschrift*, 29, p. 319-334.
5. G. Rasch (1929). Studier over Kummers Konvergenskriterium. *Matematisk Tidsskrift,* B, p. 37-49.
6. G. Rasch (1930). Om systemer af lineære Differentialligninger. *Den syvende skandinaviske Matematikerkongres i Oslo 1930.* p. 117-126.

[1] A more precise reference is: Professor, dr. phil. Georg Rasch. Oversigt over væsentligere arbejder. Udgivet i anledning af hans 70 års dag. 21. 09. 71. IMSOR 1971.(IMSOR – Institut for Matematisk Statistik og Operationsanalyse).

7. G. Rasch (1930). *Om Matrixregning og dens Anvendelser paa Differens- og Differentialligninger.* Copenhagen, pp. 115. Doctoral thesis.

8. G. Rasch (1931). Notes on the Gamma-function. *Annals of Mathematics,* second series, 32, p. 592-599.

9. G. Rasch (1934). Zur Theorie und Anwendung des Produktintegrals. *Journal für Mathematik,* 2, p. 65-119.

10. M. Fog, G. Rasch und G. Stürup (1934). Über die Resorption der Zerebrospinalflüssigkeit. *Skandinavischen Archiv für Physiologie*, 69, p. 127-150.

11. G. Rasch (1934). Om Normaltal. *Medicinske Tal*, Eds Knud Secher, p. 9-17, Copenhagen.

12. G. Rasch (1937). Om Restledsintegralet $\int_T^\infty e^{xt} dA(t)$ *Matematisk Tidsskrift*, B, p. 97-100.

13. Børge Worning (1937). *A Quantitative Röntgen-Biological Method.* Copenhagen. p. 40-48, 77-79.

14. Erik Mogensen (1938). *Studies on the Size of the Red Blood Cells.* Copenhagen. p. 26-50.

15. Th. Madsen and G. Rasch (1938). On Immunization of Rabbit Groups. *Acta Pathol. et Microbiol. Scand,* suppl. 37, p. 369-380.

16. Knud Halberg (1939). *Tuberculosis at Nakskov, I, II.* Copenhagen, p. 65-76.

17. Ottar Dybing, Klaus Hansen and G, Rasch (1939). Om visse fejlkilder ved Widmarks metode til bestemmelse af alkohol i blod. *Tidsskrift for kjemi og bergvesen*, p. 114-117.

18. E. Juel Henningsen and G. Rasch (1938). Poliomyelitisepidemien 1937. Epidemiologi og Statistik med særligt Henblik paa Reconvalescentserumterapien. II Klinisk Statistik. *Nordisk medicin*, p. 2763.

19. Sk. V. Gudjonsson (1940). *P. Carl Petersen's Fonds Ernæringsekspedition til Færøerne 1936-37.* Copenhagen. Chapter IX, XIV, XVI, XVIII, XIX.

20. Frode Westergaard (1940). *Staseprøven og dens kliniske betydning.* Copenhagen. p. 52-54.

21. G. Rasch and E. Groth-Petersen (1941). En metode til rationalisering og Simplificering af Kostberegninger. *Nordisk Medicin*, p. 1897.

22. G. Rasch (1941). Om Vurdering af fejl på kemiske målinger.I. *Kemisk Månedsblad og Nordisk Handelsblad for kemisk Industri*, p. 157-168.

23. A. Hald, M. Jersild and G. Rasch (1942). On the Determination of the Phagocytic Power of Leucocytes. *Acta path.*, XX, p. 64-85.

24. P. Færgeman and G.Rasch (1943). Diastasebestemmelser i blod og urin. *Nordisk Medicin*, p. 1347.

25. Jens L. Hansen (1943). *Om Ulcus Ventriculi S. Duodeni og Cancer Ventriculi i et Sectionsmateriale.* Copenhagen, p. 311-314.

26. Knud Rosenquist (1943). *Om Strumaproblemet.* Copenhagen. Mathematical Appendix by A. Hald and G. Rasch. p. 353-362.

27. Erik Andreasen (1943). *Studies on the Thymolymphatic System.* Copenhagen. p. 55-60.

28. A. Hald and G. Rasch (1943). *Nogle Anvendelser af Transformationsmetoden i den normale Fordelings Teori.* Festskrift til Professor dr. phil J. F Steffesen 28 Febr. 1943, p. 52-65.

29. H. Lindholm and P. Terp in cooperation with G. Rasch (1944). Nogle undersøgelser over Styrkebestemmelse af Glandula Thyreoidea Siccata og Præparater heraf, I. *Archiv for Pharmaci og Chemi*, 18. p. 1-21.

30. G. Rasch i samarbejde med H. Lindholm and P. Terp (1945). Nogle undersøgelser over Styrkebestemmelse af Glandula Thyreoidea Siccata og præparater heraf, II. *Archiv for Pharmaci og Chemi*, 5. p. 105-120.

31. Henning Vogelius (1945). *Basal Metabolism of Girls.* Copenhagen. p. 23-40, 50-74, 78-87.

32. Axel Perdrup (1945). *Om stivkrampe.* Copenhagen. p. 187-201, 202-210.

33. G. Rasch (1945). En syntese af nogle Interpolations-, Kvadratur- og Summationsformler. *Matematisk Tidsskrift*, B. p. 129-141.

34. G. Rasch (1946). A Principle for Deriving the Remainder Termes in Some Serial Expansions. *Dixiòme Congrès des Mathématiciens Scandinaves.* Copenhague 1946. p. 193-300.

35. G. Rasch anm. af H. Cramér' (1945). Mathematical Methods of Statistics. *Skand. Aktuarietidsskrift,* Uppsala, p. 127-132.

36. Jens Bing, J. Næser, G. Rasch and Kaj Røjel (1946). Serum Proteins in Normal People. *Acta Medica Scandinvica,* CXXVI, p. 351-369.

37. Harald Ewertsen (1946). *Dyreexperimentelle Undersøgelse over colibacillers Patogenitet og Effekten af coliserum.* Copenhagen, p. 142-148.

38. Erik Jacobsen, C. M. Plum and G. Rasch (1947). On the Accuracy of Reticulocyte Counts. *Acta pathologica*, XXIV, p. 554-566.

39. P. O. Petersen and G. Rasch (1947). Tandmorfologiske Undersøgelser som Grundlag for Caries-registrering. *Ordontologisk Tidsskrift*, 4-5, p. 447-471.

40. Karen Margrethe Simonsen (1947). *Examination of Children from Children's Homes and Day-Nurseries.* Copenhagen. p. 175-188.

41. C. G. Lund, E. Peülicke Nielsen and K. Pedersen-Bjergaard (1947). *The Preparation of Solutions Isoosmotic with Blood, Tears, and Tissue.* Copenhagen. p. 23-36.

42. G. Rasch (1947). *Om Vurdering af Intelligensprøver.* Københavns Universitets psykologiske Laboratorium. Dupl. An abridged version can be found in *Beretninger fra det 1. nordiske psykologmøde.*

43. G. Rasch (1947). Recent Biometric Developments in Denmark. *Biometric*, 4, p. 172-175.

44. G. Rasch (1948). A Functional Equation for Wishart's Distribution. *The Annals of Mathematical Statistics*, XIX. p. 262-266.

45. Chr. Hamburger with a statistical appendix by G. Rasch (1948). "Micro-methods" for the Determination of 17-Ketosteroids in Urine. *Acta endocrinol,* 1. p. 385-390.

46. I Scheibel, S. Tulinius, G. Rasch, K. Bojlén and Chr. Borg Petersen (1948). Immunization of Adults against Diphteria. *Acta pathologica*, XXV. p. 319-350.

47. Harald Olesen (1948). *Morfologiske sperma- og tesis-undersøgelser.* Copenhagen. p. 48-54, 132-133.

48. G. Rasch (1950). Statistik og forsøg. Orientering i et materiale af markforsøg over kvælstofgødninger. *Tidsskrift for Planteavl*, 53. p. 369-401.

49. Hans Chr. Engbæk and G. Rasch (1950). Animal Experiments with Pfeiffer's Bacillus Type B. II. Assessment of Bacteraemia. *Acta pathologica*, vol. XXVII, fasc. 6. p. 830-843.

50. G. Rasch (1950). A Vectorial t-Test in the Theory of Normal Multivariate Distributions. *Matematisk Tidsskrift,* B. p. 76-80.

51. G. Rasch (1951). Om kausal vurdering af terapi. *Ugeskrift for læger*. p. 1625-1630.

52. M. Weis Bentzon, O. Maaløe and G. Rasch (1951). An Analysis of the Mode of Increase in Number on Intracellular Phage Particles at Different Temperatures. *Acta pathologica et microbiologica Scandinavica,* XXX, p. 243-270.

53. Karsten Ibsen and G. Rasch (1951). *En formel analyse af begrebet frihedsgrader.* Københavns Universitets statistiske seminar. Dupl. pp. 16.

54. Knud Rasmussen (1951). *Undersøgelser over Tuberkelbaciller i Expectorat med særligt Henblik på deres Vækstforhold.* Copenhagen. p. 136-142.

55. Erik Hjalmar Larsen (1952). *Svingningerne i prothrombinaktiviteten hos nyfødte.* Copenhagen. p. 70-91.

56. M. Ellehammer, Carl Aage Larsen and G. Rasch (1952). *Statistiske undersøgelser over nuværende og tidligere læseklasseelever i København.* Rapport til kommissionen angående omorganisation af forsorgen for talelidende. Arbejds- og Social-ministeriet. Typewritten.

57. Georg Rasch (1952). Statistik. *Nordisk Sommeruniversitet 1951* Aarsagsproblemet. Copenhagen. p. 244-251.

58. Georg Rasch (1953). On simultaneous factor analysis in several populations. Uppsala Symposium on Psychological Factor Analysis. 17-19 March 1953. *Nordisk Psykologi's Monograph*, Series No 3 p. 65-71.

59. Georg Rasch (1953). Debate at the Uppsala Symposium on Psychological Factor Analysis. 17-19 March 1953. *Nordisk Psykologi's Monograph Series No 3.* p. 76-77, 77-79, 82, 83-84, 84-85, 86-87, 88, 90.

60. G. Rasch (1954). *Statistisk analyse af resultaterne af forsøgsrække D II.* Rapport over forsøg vedr, oliefyr til centralvarmeanlæg. Copenhagen. p. 111-126.

61. G. Rasch, M. W. Bentzon (1956). *Statistical evaluation of pilot experiment results.* World Health Organization 1956 (WHO/VDT/SERO/20). Dupl.

62. Kn. Bojlén, G. Rasch and M. Weis Bentzon (1954). The Age Incident of the Menarche in Copenhagen. *Acta Obst. et Gyn. Scandinav.,* 33, p. 405-433.

63. G. Rasch (1954). *Contributions to the Theory of Stochastic Differential Equations.* Københavns Universitets statistiske seminar.

64. G. Rasch (1955). Statistisk belysning af jorden og arbejdskraftens produktivitet. Virkelighed og beskrivelse. *Nordisk Sommeruniversitet, Otnäs 1954.* Copenhagen. p. 167-172.

65. G. Rasch (1955). *Om populationer, standarder og profiler.* 6. nordiske militærpsykologkonference. Copenhagen 9-11 September 1954. Militærpsykologisk arbejdsgruppe. Copenhagen. Dupl. p. 47-75.

66. G. Rasch (1955). *Standpunktsprøverne i 5. klasserne i Stor-København 1955. Forcensur.* Prøveudvalget. Copenhagen. Dupl.

67. G. Rasch (1955). *Forsøget med standpunktsprøverne i dansk i 5. klasser i Stor-København april-maj 1954.* Rapport. Prøveudvalget. Copenhagen. Dupl.

68. C. H. Gudnason and G. Rasch (1956). *Undersøgelser over usikkerheden på bestemmelsen af røgens temperatur og CO_2-indhold i aftræksrør på centralvarmekedler.* Rapport til Teknologisk Instituts varmetekniske afdeling. Copenhagen. Dupl.

69. C. H. Gudnason and G. Rasch (1956). *Undersøgelser over fyringsøkonomien ved centralvarmeanlæg i praksis.* Rapport til Teknologisk Instituts varmetekniske afdeling. Copenhagen. Dupl.

70. P. Krag, M. W. Bentzon, G. Rasch (1956). *Freeze-dried sera from syphilitic and non-syphilitic donors, 30 sera collected 1952-1953. Final statistical evaluation.* World Health Organization 1956 (WHO/VDT/SERO/63). Dupl.

71. G. Rasch (1957). A Method of Indirect Measurement in Productivity Studies I/II. *Productivity Measurement Review*, 10, p. 23-44; n. 11, p. 42-68.

    71.a. G. Rasch (1957). Une méthode de mesure indirecte dans les études de productivité. *Revue de la Mesure de la Productivité.*, 10, p. 25-47; 11, p. 45-71.

72. O. Scherbaum, and G, Rasch (1957). Cell Size Distribution and Single Cell Growth in Tetrahymena Pyriformis Gl. *Acta pathologica et microbiologica scandinavica,* XLI, p. 161-182.

73. Georg Rasch (1957). *Rapport over statistisk analyse af forsøg over trykfaldet gennem nogle ventiler og haner, I/II.* Rapport til Teknologisk Instituts varmetekniske afdeling. Copenhagen. Dupl.

74. F. Abildgaard Jørgensen (1957). *Kompendium af statistik for psykologer og pædagoger.* Printed in Copenahgen. pp. 65.

75. Kaj Arentsen (1957). An investigation of the questionnaire method by means of the Cornell Index (Form N2). I *Acta Psychiatrica et Neurologica Scand.*, 32. p. 231-256. p. 336-240.

76. Georg Rasch (1957). *Nogle studier over statistiske metoder til analyse af tidsmæssige variationer i antal beskæftigede og ledige medlemmer at de forskellige byggefag og i antallet af godkendte, påbegyndte og fuldførte byggerier.* Rapport til Arbejdsmarkedskommissionens sekretariat. Dupl.

77. Georg Rasch (1957). *Om en mekanisme der frembringer en tidsrække fra en anden.* Bilag til rapport til Arbejdsmarkedskommissionens sekretariat. Dupl. pp. 33.

78. Georg Rasch (1958). *Om anvendelse af et almindeligt måleprincip til brobygning mellem ensartede psykologiske prøver.* Danmarks pædagogiske Institut. Copenhagen. Dupl.

79. Børge Prien and Georg Rasch (1958). *Standardisering af en prøve i regning og matematik (RMP-53) og en diskussion af standardiseringens principielle grundlag* (with an english summary)I/II Militærpsykologisk Tjeneste, Copenhagen.Dupl. pp. 107.

80. Georg Rasch (1959). *Elementær matematik for psykologistuderende.* Copenhagen. pp. 76.

81. Georg Rasch, J, Ludvigsen and Grete Thorbek (1958). *Evaluation of errors in determination of N, P and Ca in feeds, feces and urine.* Proc, I. Symposium on Energy Metabolism. Copenhagen. September 1958. Publ. no. 8 from European Association for Animal Production. p. 163-181.

82. Georg Rasch (1958). *Kan dispositioner for ulykkestilfælde måles?* Københavns Universitets statistiske seminar. Dupl.

83. Georg Rasch (1959). Processer og udvikling. *Nordisk Sommeruniversitet 1958: Udviklingsproblemer.* Copenhagen. p. 86-96.

84. Georg Rasch (1959). Om vurderingens indslag i erkendelsen. *Nordisk Sommeruniversitet 1959: Erkendelse, Vurdering og Valg.* Fire indledningsforelæsninger af Alf Ross, Georg Rasch, Mogens Blegvad og Mogens Pihl. Copenhagen. p. 23-33.

85. Georg Rasch (1960). *Probabilistic Models for Some Intelligence and Attainment Tests.* Studies in Mathematical Psychology I. Danmarks pædagogiske Institut. Copenhagen.

86. Georg Rasch (1961). On general laws and the meaning of measurement in psychology. *Proc. 4 Berkeley Symposium on Mathematical Statistics and Theory of Probability,* 20/6-30/7 1960. Vol. IV. p. 321-333. Berkeley: University of California Press.

87. Georg Rasch (1962). Om karaktergivningsproblemet. *Dansk Pædagogisk Tidsskrift*, 1. p. 1-9.

88. Ulf Christiansen (1963). *Referat af professor G. Rasch's forelæsninger over statistikkens teori. Almen teori.* Universitetets Statistiske Institut. 1962-63. Dupl. Printed in 1966, Copenhagen.

89. Ulf Christiansen (1963). *Referat af professor G. Rasch's forelæsninger over statistikkens teori. Målingsmodeller.* Universitetets Statistiske Institut. 1962-63. Dupl. Printed in 1966.

90. Rasch, Georg (1964). Om forholdet mellem praktisk og teoretisk statistik, *10. Nordiske statistikermøde, Copenhagen 1964.* p. 72-80.

91. Georg Rasch (1965). *Målingsmodellerne og deres principielle baggrund.* Københavns Universitet Institut for Matematisk Statistik og Universitetets Statistiske Institut. Referat af G. Rasch's forelæsningsrække i det statistiske seminar.

92. Georg Rasch (1965). *An individual-centered approach to item analysis with two categories of answers.* Dupl. pp. 25.

93. Georg Rasch (1966). An Individualistic Approach to Item Analysis. *Readings in mathematical social science*, Lazersfeld and N. W. Henry eds. pp. 89-108. Chicago Science Research Associates.

94. Georg Rasch (1966). An item analysis which takes individual differences into account. *BR. J. of Math. and Stat. Psych.*, 19. p. 49-57.

95. Georg Rasch (1967). An informal report of objectivity in comparisons. *Psychological measurement theory, Proceedings of the NUFFIC International summer Session in Science at "Het Oude Hof" Den Haag. Juli 14-28, 1966.* Leiden.

96. Georg Rasch (1968). *En reanalyse af danske og svenske forsøg over virkningen af hastighedsbegrænsninger på trafikulykker.* Copenhagen. Dupl.

97. Georg Rasch (1968). A Mathematical Theory of objectivity and Its Consequences for Model Construction. *European Meeting on Statistics, Econometric and Management Sciences.* Amsterdam 2-7 September 1968. pp. 31.

98. Ulf Christiansen and Jon Stene (1968). *G. Rasch's lærebog i statistik, bind 1.* Copenhagen. pp. 325.

99. Ulf Christiansen and Jon Stene (1969). *G. Rasch's lærebog i statistik, bind 2.* Copenhagen. pp.s 325.

100. Georg Rasch (1972). Objektivitet i Samfundsvidenskaberne et metodeproblem. *Nationaløkonomisk tidsskrift.*

101. Georg Rasch. On specific objectivity (1977). An attempt at formalizing the request for generality and validity of scientific statements. *Danish Yearbook of Philosophy*, 14, 58-94.