# *16. PARAMETER ESTIMATION*

Parameter estimation for Rasch measurement is usually done by a computer program like BICAL (Wright & Mead, 1976), BIGSCALE (Wright, Linacre & Schulz, 1990) or BIGSTEPS (Wright & Linacre, 1997).

The PROX procedure, however, is a method of estimation so easy to apply that it is completely manageable by hand. The simplicity of PROX is useful because it details exactly how the Rasch model works in practice. PROX accomplishes the primary aims of Rasch item analysis:

1) linearization of item raw scores (P-values) onto an interval scale with relevant errors of calibrations and

2) adjustment for the sampling effects of person ability.

In so doing, PROX almost always approximates the results obtained by more elaborate procedures extremely well.

The simplification which enables PROX is to approximate the effects on item calibration of sample ability with a sample mean and standard deviation and the effects on person measurement of test item difficulty with a test mean and standard deviation. This simplification makes PROX easy to apply by hand. Nothing more than the observed distributions of item and persons scores, a hand calculator and pencil and paper are needed.

PROX is as applicable to large assessment problems like national item banking as it is to evaluating a small classroom of examinees. The PROX algorithm is the working basis of most successful computer assisted testing (CAT) programs.

In this chapter we

1) outline the PROX equations, and

2) explain how these equations implement Rasch measurement.

The chapter can be used to guide the user in calibrating items and measuring persons from their own data. An example worked out in numerical detail can be found in the second chapter of *Best Test Design* (Wright & Stone, 1979, pp. 28-45).

## THE PROX ESTIMATION EQUATIONS

PROX simplifies the representation of person abilities $b_n$ to a normal distribution with sample mean ability $M$ and sample ability standard deviation $\sigma$ and the representation of item difficulties $d_i$ to a normal distribution with test item mean difficulty $H$ and test item difficulty standard deviation $w$.

When that is done, then the measure $b_n$ for person $n$ with person score $r_n$ on a test of $L$ items becomes

$$b_n = H + X \log[r_n / (L - r_n)] \qquad 16.1$$

where

$H$ = mean difficulty of the $L$ items taken,
$X$ = the scaling necessary to adjust for the difficulty standard deviation $w$ of these $L$ items, $r$ = the raw test score of person $n$,

and the calibration $d_i$ for item $i$ with item score $s_i$ from a sample of $N$ persons is

$$d_i = M + Y \log[(N - s_i) / s_i] \qquad 16.2$$

where $d_i = M + Y \log[(N - s_i) / s_i]$
$M$ = mean ability of the $N$ persons taking the test,
$Y$ = the scaling necessary to adjust for the ability standard deviation $\sigma$ of these $N$ persons,
$s_i$ = the raw sample score of item $i$.

and

$$X = \left[1 + (w^2 / 2.89)\right]^{1/2} \cong \left[1 + (w^2 / 5.8)\right] \qquad 16.3$$

$$Y = \left[1 + (\sigma^2 / 2.89\right]^{1/2} \cong \left[1 + (\sigma^2 / 5.8)\right]. \qquad 16.4$$

The divisor $2.89 = 1.7^2$ comes from the scaling factor 1.7 which, because the logistic ogive for values of $1.7z$, is never more than one percent different from the normal ogive for values of $z$, brings the cumulative logistic distribution into approximate coincidence with the cumulative normal distribution. PROX uses this coincidence to obtain its simplification.

The estimates $b_n$ and $d_i$ have standard errors

$$SE(b_n) = X[L / r_n(L - r_n)]^{1/2} \cong 2.5 / L^{1/2} \qquad 16.5$$

$$SE(d_i) = Y[N / s_i(N - s_i)]^{1/2} \cong 2.5 / N^{1/2} . \qquad 16.6$$

## APPLYING THE PROX ESTIMATION EQUATIONS

This estimation method can be applied to observed item scores $s_i$ by calculating the sample score logit of item $i$ as

$$x_i = \log[(N - s_i) / s_i] \qquad 16.7$$

and to the observed person scores $r_n$ by calculating the test score logit of person $n$ as

$$y_n = \log[r_n / (L - r_n)] \qquad 16.8$$

The scaling coefficients $X$ and $Y$ can be estimated from

$$X = \{[1 + (U / 2.89)] / [1 - (UV / 8.35)]\} \qquad 16.9$$

for the person logit scaling coefficient and

$$Y = \{[1 + (V / 2.89)] / [1 - (UV / 8.35)]\} \qquad 16.10$$

for the item logit scaling coefficient.

Where $U = \left(\sum_i^L x_i^2 - Lx_.^2\right) / (L-1)$ and $8.35 = 2.89^2 = 1.7^4$,

$$U = \left(\sum_i^L x_i^2 - Lx_.^2\right) / (L-1) \qquad 16.11$$

is the item logit variance,

$$V = \left(\sum_n^N y_n^2 - Ny_.^2\right) / (N-1) \qquad 16.12$$

is the person logit variance, $x_. = \sum_i^L x_i / L$ is the item logit mean, and $y_. = \sum_n^N y_n / N$ is the person logit mean.

To complete the estimation, we anchor the scale "zero" at the center of the test by defining $H \equiv 0$ so that

$$d_i = N| + Yx_i = Y(x_i - x_.) \text{ because } M = -Yx_. \qquad 16.13$$

for each item difficulty and

$$b_n = H + Xy_n = Xy_n \text{ because } H \equiv 0 \qquad 16.14$$

for each person ability.

STANDARD ERRORS

The standard errors of these person and item estimates are

$$SE(b_n) = X[L / r_n(L - r_n)]^{1/2} \cong 2.5 / L^{1/2} \qquad 16.15$$

and

$$SE(d_i) = Y[N / s_i(N - s_i)]^{1/2} \cong 2.5 / N^{1/2} \; . \qquad 16.16$$

## SAMPLE STATISTICS

The estimates of the person sample mean $M$ and standard deviation $\sigma$ are

$$M \approx -Yx_{\bullet} \qquad 16.17$$

$$\sigma \approx 1.7(Y^2 - 1)^{1/2} \qquad 16.18$$

## ANALYSIS OF RESIDUALS

When we have estimated $b_n$ and $w$ we can use them to obtain the difference between the model's prediction and the data observed. Residuals from the model are calculated by estimating from $b_n$ and $d_i$ the model expectation at each response $\chi_{ni}$ and the subtracting this expectation from the $\chi_{ni} = 0$ or 1 which was actually observed.

The model expectation for $\chi_{ni}$ is $\Pi_{ni}$ where the Rasch model for $\Pi_{ni}$ is $\Pi_{ni} = \exp(B_n - D_i) / [1 + \exp(B_n - D_i)]$ and $B_n$ and $D_i$ are the parameters which $b_n$ and $d_i$ estimate.

The standardized residual from expectation is

$$z_{ni} = (\chi_{ni} - \Pi_{ni}) / [\Pi_{ni}(1 - \Pi_{ni})]^{1/2} \ . \qquad 16.19$$

When the data fit the model this standardized residual is distributed with mean zero and variance one. Although the expected sampling distribution of $D_i$ is not normal, we have found that values below -2 and above +2 are useful as indicators of noteworthy misfit.

We estimate $\Pi_{ni}$ from

$$P_{ni} = \exp(b_n - d_i) / [1 + \exp(b_n - d_i)] \qquad 16.20$$

where $b_n$ and $d_i$ are the estimates of $B_n$ and $D_i$, and then use the sampling distributions for $z_{ni} \approx (\chi_{ni} - P_{ni}) / [P_{ni}(1 - P_{ni})]^{1/2}$ of $z_{ni} \sim N(0,1)$ and $z_{ni}^2 \sim X_1^2$ as guidelines for evaluating the extent to which any particular set of data is sufficiently coherent to construct useful measurement.

## PERSON FIT

To measure the validity of a person's performance, calculate the sum of squared residuals $z_{ni}^2$ for that person. When the person's behavior is useful for measurement because their response pattern fits the measurement model, then their sum of square standard residuals will approximate a chi-square statistic

$$\sum_{i}^{L} z_{ni}^2 = C_n^2 \sim X_{fn}^2 \qquad 16.21$$

with degrees of freedom

$$f_n = (L-1)(N-1)/N \qquad 16.22$$

and a mean square statistic

$$V_n = C_n^2 / f_n \sim F_{fn,\infty} \qquad 16.23$$

with degrees of freedom $f_n$ and $\infty$ .

ITEM FIT

To measure the validity of an item's usage calculate the sum of squared residuals $z_{ni}^2$ for that item. When the item is useful for measurement because the pattern of its responses fits the measurement model, then its sum of squared standard residuals will approximate a chi-square statistic

$$\sum_{n}^{N} z_{ni}^2 = C_i^2 \sim X_{fi}^2 \qquad 16.24$$

with degrees of freedom

$$f_i = (N-1)(L-1)/L \qquad 16.25$$

and a mean square statistic

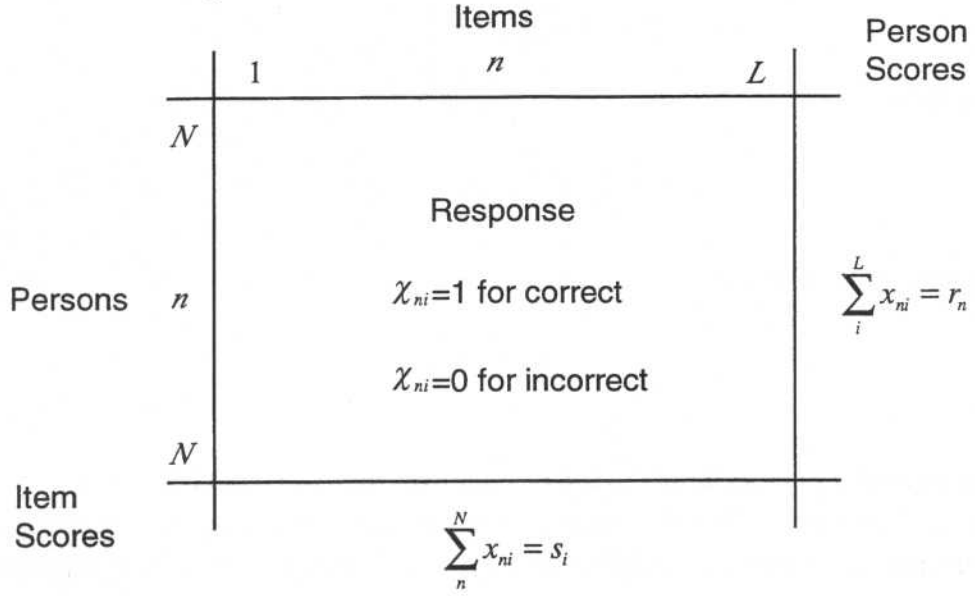
$$U_i = C_i^2 / f_i \sim F_{fi,\infty} \qquad 16.26$$

with degrees of freedom $f_i$ and $\infty$.

Now we will reexamine the details of PROX to discover what is accomplished in its application. The PROX formula enables a simple and intuitive approach to understanding how Rasch item calibration and person measurement work.

## THE RESPONSE MATRIX

Consider the response matrix:




When there is no missing data* so that this response matrix is complete, then every row total gives a person score for the same set of items and every column total gives an item score for the same set of persons. These item scores are reported as "P-values" in traditional item analysis. "P-values" are calculated by dividing the item score $s_i$ by the number of persons $N$ so that the P-value for item $i$ becomes $P_i = S_i / N$, the proportion of correct responses to item $i$.

If the person raw scores $r_n$ and item P-values $P_i = S_i / N$ were linear objective quantifiers of person ability and item difficulty, our work would seem done at this point. Indeed, person raw scores and item P-values are as far as traditional person measurement and item analysis go. As a consequence, most of the research results in educational measurement have been limited to what little can be done when raw counts of right answers are mistaken for measures.

As measures, however, raw counts have two serious drawbacks.

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* In the application of PROX (or any other Rasch model procedure), the data need not be complete. Missing data can be accommodated without trouble. This happens because the measurement structure specified by the model needs only enough data to identify a finite estimate for each person and each item. As a result none of the estimation procedures need complete data to obtain good estimates.

## RAW SCORES ARE NON-LINEAR

Raw counts are bounded in range between none right and all right. Because of this they cannot represent abilities or difficulties on a linear (interval) scale. But, if they are not linear, then the results of the arithmetic used in statistical analysis become misleading. In order to enable the substantial benefits of statistical analysis, we must transform the non-linear scores into linear measures.

## RAW SCORES ARE TEST AND SAMPLE DEPENDENT

Each observed response is the result of a person of some ability attempting an item of some difficulty. Because of this, the magnitudes of item scores $s_i$ and P-values $P_i$, which are summed over persons, depend on the particular abilities of this particular sample of persons and so are sample dependent.

The magnitudes of person scores $r_n$, which are summed over items, depend, in turn, on the particular difficulties of this particular set of items and so are test dependent.

In order to make general use of the information about person measures and item difficulties which is contained in the test item response data, we must liberate the numerical representations of person measures and item difficulties from the local effects they have on one another. We must construct objective person measures which are test-free and objective item calibrations which are sample-free.

## CONVERTING NON-LINEAR, LOCALLY DEPENDENT ITEM AND PERSON SCORES TO LINEAR, INDEPENDENT ITEM CALIBRATIONS AND PERSON MEASURES

The way the Rasch model linearizes raw scores and frees them of sample and test dependency can be seen in the following two PROX formulae:

For Items:

$$\log\left[\frac{(1 - P_i)}{P_i}\right] * \left[1 + \frac{\sigma^2}{2.7}\right]^{1/2} + M \Rightarrow d$$

[log odds linearize item P-values] [scales out the sample variance $\sigma^2$ ] [adjusts out the sample mean $H$] $\Rightarrow$ [Test-freed item calibration]

For Persons:

$$\log\left[\frac{r_n}{(1 - r_n)}\right] * \left[1 + \frac{w^2}{2.7}\right] + H \Rightarrow b_n$$

[log odds linearize person raw scores] [scales out the test item variance $w^2$ ] [adjusts out the test item mean $H$] $\Rightarrow$ [Test-freed person measurement]

For more detail see the example of PROX item calibration and person measure in Chapter 2 of Best Test Design (Wright & Stone, 1979, pp. 28-45).

# MEASUREMENT ESSENTIALS

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