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COMBINATORIAL PROBLEMS IN A CLASS OF STATISTICAL MODELS

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The present paper does not pretend to offer substantially new results. The background for delivering it is that when building up a rather general class of models and attempting to exploit them in statistical practice, I ran into a class of combinatorial coefficients which resisted my efforts. But not being much experienced in combinatorics I may very well have overlooked even easy ways out. When now submitting my problem and my tentative approaches to it for discussion, I hope to get some orientation in the matter.

1. The main difficulties already show in the simplest case which arise in connection with intelligence testing in psychology.

A large number of persons are exposed to some 2o questions to each of which the answer is either correct (+) or non-correct (-). For reasons discussed elsewhere I assume the probability that the answer of person no. ν to item no. i is correct to be of the form

$$(1.1) \qquad p\{+|\nu,i\} = \frac{\xi_\nu \varepsilon_i}{1+\xi_\nu \varepsilon_i}, \quad \begin{matrix}\nu=1,\dots,n\\ i=1,\dots,k\end{matrix}$$

where ξ_ν and ε_i are parameters characterizing the persons (their "abilities") and the items (their "easiness"), respectively. Writing

$$a_{\nu i} = \begin{cases} 1 & \text{for +answer} \\ 0 & \text{for -answer} \end{cases}$$

the model may be written

$$(1.2) \qquad p\{a_{\nu i}\} = \frac{(\xi_\nu \varepsilon_i)^{a_{\nu i}}}{1+\xi_\nu \varepsilon_i} \quad .$$

Assuming stochastical independence between the answers of the persons to the items the joint probability of the whole matrix of answers

$$A = (a_{\nu i})$$

becomes

$$(1.3) \qquad p\{A\} = \frac{\prod_{(\nu)} \xi_\nu^{a_{\nu o}} \prod_{(i)} \varepsilon_i^{a_{oi}}}{\prod_{\nu} \prod (1+\xi_\nu \varepsilon_i)}$$

where

$$a_{\nu o} = \sum_{i=1}^{k} a_{\nu i} \quad , \quad a_{oi} = \sum_{\nu=1}^{n} a_{\nu i}$$

are the total numbers of +answers for each person and to each item, respectively.

With the notations

$$\mathcal{A}_{*o} = (a_{1o},\ldots,a_{no}) \quad , \quad \mathcal{A}_{o*} = (a_{o1},\ldots,a_{ok})$$

$$\Xi = (\xi_1,\ldots,\xi_n) \quad , \quad \mathcal{E} = (\varepsilon_1,\ldots,\varepsilon_k)$$

$$\Xi^{\mathcal{A}_{*o}} = \xi_1^{a_{1o}} \ldots \xi_n^{a_{no}} \quad , \quad \mathcal{E}^{\mathcal{A}_{o*}} = \varepsilon_1^{a_{o1}} \ldots \varepsilon_k^{a_{ok}}$$

$$\gamma(\Xi^* \varepsilon) = \prod_{(\nu)} \prod_{(i)} (1+\xi_\nu \varepsilon_i)$$

(1.3) is simplified to

$$(1.4) \qquad p\{A\} = \frac{\Xi^{\mathcal{A}_{*o}} \varepsilon^{\mathcal{A}_{o*}}}{\gamma(\Xi^* \varepsilon)}$$

Obviously the joint distribution function of the exponents is

$$(1.5) \qquad p\{\mathcal{A}_{*o}, \mathcal{A}_{o*}\} = \begin{bmatrix} \mathcal{A}_{*o} \\ \mathcal{A}_{o*} \end{bmatrix} \frac{\Xi^{\mathcal{A}_{*o}} \varepsilon^{\mathcal{A}_{o*}}}{\gamma(\Xi^* \varepsilon)}$$

where the bracket symbol represents the number of (0,1)-matrices with the marginal sums $(a_{\nu o})$ and (a_{oi}). On dividing (1.5) into (1.4) we get the conditional probability of the set of answers, given the marginals:

$$(1.6) \qquad p\left\{A \mid \mathcal{A}_{*o}, \mathcal{A}_{o*}\right\} = \frac{1}{\begin{bmatrix} \mathcal{A}_{*o} \\ \mathcal{A}_{o*} \end{bmatrix}}$$

which is independent of the parameters. Thus $\mathcal{A}_{*o}$ and $\mathcal{A}_{o*}$ are sufficient estimators of Ξ and $\mathcal{E}$.

Furthermore, summing over $\mathcal{A}_{o*}$ in (1.5) we obtain the marginal distribution

$$(1.7) \qquad p\left\{\mathcal{A}_{*o}\right\} = \frac{\Xi^{\mathcal{A}_{*o}}\,\gamma_{\mathcal{A}_{*o}}(\varepsilon)}{\gamma(\Xi^{*}\varepsilon)}$$

where

$$(1.8) \qquad \gamma_{\mathcal{A}_{*o}}(\varepsilon) = \sum_{(\mathcal{A}_{o*})} \begin{bmatrix} \mathcal{A}_{*o} \\ \mathcal{A}_{o*} \end{bmatrix} \varepsilon^{\mathcal{A}_{o*}}$$

and on dividing this into(1.5) we get the probability of the item totals, given the personal totals:

$$(1.9) \qquad p\left\{\mathcal{A}_{o*} \mid \mathcal{A}_{*o}\right\} = \begin{bmatrix} \mathcal{A}_{*o} \\ \mathcal{A}_{o*} \end{bmatrix} \frac{\varepsilon^{\mathcal{A}_{o*}}}{\gamma_{\mathcal{A}_{*o}}(\varepsilon)}$$

which only depends on the item parameters. Symmetrically we have

$$(1.1o) \qquad p\left\{\mathcal{A}_{*o} \mid \mathcal{A}_{o*}\right\} = \begin{bmatrix} \mathcal{A}_{*o} \\ \mathcal{A}_{o*} \end{bmatrix} \frac{\Xi^{\mathcal{A}_{*o}}}{\gamma_{\mathcal{A}_{o*}}(\Xi)}$$

The decisive feature in the formulae(1.9) and (1.1o) is that in principle they render it possible to estimate and otherwise appraise the item parameters independently of the person parameters, and the other way round.

We may even proceed a step further. Considering two groups of persons, selected in any way desired - no randomization being required - we wish to test the null hypothesis that the same ε's apply to both groups. Under this hypothesis the joint conditional probability of the two sets of totals for items $\mathcal{A}_{o*}^{(1)}$ and $\mathcal{A}_{o*}^{(2)}$, given the two sets of totals for persons $\mathcal{A}_{*o}^{(1)}$ and $\mathcal{A}_{*o}^{(2)}$, is

(1.11)
$$p\left\{\mathcal{A}_{o*}^{(1)}, \mathcal{A}_{o*}^{(2)} \mid \mathcal{A}_{*o}^{(1)}, \mathcal{A}_{*o}^{(2)}\right\} = \begin{bmatrix}\mathcal{A}_{*o}^{(1)}\\ \mathcal{A}_{o*}^{(1)}\end{bmatrix}\begin{bmatrix}\mathcal{A}_{*o}^{(2)}\\ \mathcal{A}_{o*}^{(2)}\end{bmatrix}\frac{\varepsilon^{\mathcal{A}_{o*}^{(1)}+\mathcal{A}_{o*}^{(2)}}}{\gamma_{\mathcal{A}_{*o}^{(1)}}(\varepsilon)\gamma_{\mathcal{A}_{*o}^{(2)}}(\varepsilon)},$$

Now pools the two groups. Clearly the vector of the totals for the persons is just

$$\mathcal{A}_{*o} = (\mathcal{A}_{*o}^{(1)}, \mathcal{A}_{*o}^{(2)}) \quad ,$$

and any possible vector of totals for items $\mathcal{A}_{o*}$ must be the sum of two possible vectors, one for each group:

$$\mathcal{A}_{o*} = \mathcal{A}_{o*}^{(1)} + \mathcal{A}_{o*}^{(2)} = \mathcal{A}_{*o}^{(o)}$$

Therefore our bracket symbols must satisfy the addition rule

$$(1.12) \qquad \begin{bmatrix}(\mathcal{A}_{*o}^{(1)}, \mathcal{A}_{*o}^{(2)})\\ {}^{(o)}_{o*}\end{bmatrix} = \sum \begin{bmatrix}\mathcal{A}_{*o}^{(1)}\\ \mathcal{A}_{o*}^{(1)}\end{bmatrix}\begin{bmatrix}\mathcal{A}_{*o}^{(2)}\\ \mathcal{A}_{o*}^{(2)}\end{bmatrix}$$

where the summation extends over $\mathcal{A}_{o*}^{(1)}$'s and $\mathcal{A}_{o*}^{(2)}$'s with the fixed sum $\mathcal{A}_{o*}^{(o)}$. Accordingly also

$$(1.13) \qquad \gamma_{\mathcal{A}_{*o}}(\varepsilon) = \gamma_{\mathcal{A}_{*o}^{(1)}}(\varepsilon)\,\gamma_{\mathcal{A}_{*o}^{(2)}}(\varepsilon) \quad .$$

Thus

$$(1.14) \qquad p\left\{\mathcal{A}_{o*}^{(o)} \mid (\mathcal{A}_{*o}^{(1)}, \mathcal{A}_{*o}^{(2)})\right\} = \begin{bmatrix}(\mathcal{A}_{*o}^{(1)}, \mathcal{A}_{*o}^{(2)})\\ \mathcal{A}_{o*}^{(o)}\end{bmatrix}\frac{\varepsilon^{\mathcal{A}_{o*}^{(o)}}}{\gamma_{\mathcal{A}_{*o}}(\varepsilon)}$$

divided into (1.11) gives

$$(1.15) \qquad p\left\{\mathcal{A}_{o*}^{(1)}, \mathcal{A}_{o*}^{(2)} \mid \mathcal{A}_{o*}^{(o)}, \mathcal{A}_{*o}\right\} = \frac{\begin{bmatrix}\mathcal{A}_{*o}^{(1)} & \mathcal{A}_{*o}^{(2)}\\ \mathcal{A}_{o*}^{(1)} & \mathcal{A}_{o*}^{(2)}\end{bmatrix}}{\begin{bmatrix}(\mathcal{A}_{*o}^{(1)}, \mathcal{A}_{*o}^{(2)})\\ \mathcal{A}_{o*}^{(o)}\end{bmatrix}}$$

In consequence the test aimed at is independent of all parameters, the ε's included.

At this point you may recall R.A. Fisher's so called exact test for comparing relative frequencies governed by binomial laws.

It leads to a non-parametric hypergeometric distribution

$$(1.16) \qquad p\left\{a_1, a_2 | a_o, n_1, n_2\right\} = \frac{\binom{n_1}{a_1}\binom{n_2}{a_2}}{\binom{n_o}{a_o}}$$

Here, however, we are fairly well off. For one thing the mean value and other moments of a_1 for given a_o are readily available, e.g.

$$\mathfrak{M}\{a_1 | a_o, n_1, n_2\} = \frac{a_o n_1}{n_o}$$

Moreover, $\log n!$ being rather extensively tabulated and $\log \binom{n}{a}$ to some extent, and on top of that good approximations being available, it is often an easy task to compute the single probabilities (1.16) and even sums of them.

But what about (1.15)? Could we find explicit - and manageable - expressions for $\mathfrak{M}\{\mathcal{A}_{o*}^{(1)} | \mathcal{A}_{o*}^{(o)}, \mathcal{A}_{*o}\}$? Or is anything else known, for instance asymptotic properties which would make the computation of test probabilities feasible ?

Of course the addition rule (1.12) may be helpful. In particular we may notice that if the second group comprises one person only, with r correct answers, then the coefficients $\begin{bmatrix} r \\ \mathcal{A}_{o*}^{(2)} \end{bmatrix}$ vanish except when $\mathcal{A}_{o*}^{(2)}$ consists of 1 in r places and 0's elsewhere in which cases $\begin{bmatrix} r \\ \mathcal{A}_{o*}^{(2)} \end{bmatrix} = 1$. Thus we get a simple recurrence formula

$$\begin{bmatrix} (\mathcal{A}_{*o}^{(1)}, r) \\ \mathcal{A}_{o*} \end{bmatrix} = \sum \begin{bmatrix} \mathcal{A}_{*o}^{(1)} \\ \mathcal{A}_{o*} - \mathcal{E}_{i_1, \ldots, i_r} \end{bmatrix}$$

where $\mathcal{E}_{i_1, \ldots, i_r}$ stands for the said structures of $\mathcal{A}_{o*}^{(2)}$ and the summation extends over the field $1 \leqq i_1 < \ldots < i_r \leqq k$.

By means of this recurrence formula it is easy to compute the bracket symbols in case of a few persons. But in most data available the number of persons ranges from 1oo to 1ooo or more.

Generating functions may also be helpful. There are at least two of them.

According to (1.5) we have

$$(1.17) \qquad \gamma(\Xi^{*}\mathcal{E}) = \prod\prod(1+\xi_\nu\varepsilon_i) = \sum \begin{bmatrix}\mathcal{A}_{*o}\\ \mathcal{A}_{o*}\end{bmatrix} \Xi^{\mathcal{A}_{*o}}\, \mathcal{E}^{\mathcal{A}_{o*}}$$

for arbitrary Ξ and $\mathcal{E}$. Increasing the number of persons and/or the number of items is suggestive, but leads to no more than the addition rule (1.12), I think. (1.8) is another generating function as the left hand term may be evaluated by means of (1.13). In fact, we may split up the group of persons considered according to $a_{\nu o}$. Denote the number of persons with $a_{\nu o}=r$ by $c_r (r=0,1,\dots,k)$ then it follows from (1.13) that

$$(1.18) \qquad \gamma_{\mathcal{A}_{*o}}(\mathcal{E}) = \gamma_0^{c_0}(\mathcal{E})\ \gamma_1^{c_1}(\mathcal{E})\dots\gamma_k^{c_k}(\mathcal{E})$$

where $\gamma_r(\mathcal{E})$ is defined by (1.8) for one person. In that case $\mathcal{A}_{o*}$ is of the form $\mathcal{b}_{i_1,\dots,i_r}$ mentioned above, so $\gamma_r(\mathcal{E})$ is the elementary symmetrical function of r'th order for $\varepsilon_1,\dots,\varepsilon_k$, defined by

$$(1.19) \qquad \sum_{r=0}^{k} \gamma_r(\mathcal{E})x^r = \prod_{i=1}^{k}(1+\varepsilon_i x) \quad .$$

In particular $\gamma_0(\mathcal{E})=1$.

If the coefficients, in the development of $\gamma_r^c(\mathcal{E})$, were known or readily computable, a composition in accordance with (1.18) might be within reach for not too large k. But is anything known about these coefficients ? What, for instance, happens as $c\to\infty$?

2. In general more than two responses are available, but in many practical cases the number is finite. In that case the model (1.1) is readily extended to

$$(2.1) \qquad p\left\{x^{(\mu)}\mid\nu,i\right\} = \frac{\xi_\nu^{(\mu)}\ \varepsilon_i^{(\mu)}}{\Xi_\nu\varepsilon_i^{*}}\ , \quad \begin{matrix}\Xi_\nu = (\xi^{(1)},\dots,\xi^{(m)})\\ \varepsilon_i = (\varepsilon^{(1)},\dots,\varepsilon_i^{(m)})\end{matrix}$$

in which we have for each possible answer $x^{(\mu)}$ assigned a parameter $\xi_\nu^{(\mu)}$ to any person and a parameter $\varepsilon_i^{(\mu)}$ to any item.

Denote by $\mathcal{A}_{\nu i}$ the selection vector $(0,\ldots,1,\ldots,0)$ with $m-1$ 0-elements and a 1 at the μ-th place if $x^{(\mu)}$ were the observed answer of person ν to item i. Then the model may be recast into a form analogous to (1.2):

$$p\left\{\mathcal{A}_{\nu i}\right\} = \frac{\Xi_\nu^{\mathcal{A}_{\nu i}}\, \varepsilon_i^{\mathcal{A}_{\nu i}}}{\Xi_\nu\, \varepsilon_i^*} \quad , \tag{2.2}$$

and the whole argument runs perfectly similar to the previous case.

Thus introducing the parameter matrices

$$\Xi = \begin{pmatrix} \Xi_1 \\ \cdots \\ \Xi_n \end{pmatrix} \quad , \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \cdots \\ \varepsilon_k \end{pmatrix}$$

and the matrix build up by the totals of the selection vectors per person and per item

$$A_{*o} = \begin{pmatrix} \mathcal{A}_{1o} \\ \cdot\;\cdot\;\cdot \\ \mathcal{A}_{no} \end{pmatrix} \quad , \quad A_{o*} = \begin{pmatrix} \mathcal{A}_{o1} \\ \cdot\;\cdot\;\cdot \\ \mathcal{A}_{ok} \end{pmatrix}$$

we get for the matrix whose elements are the selection vectors observed

$$A = (\mathcal{A}_{\nu i})$$

the probability

$$p\left\{A\right\} = \frac{\prod\limits_{(\nu)} \Xi_\nu^{\mathcal{A}_{\nu o}} \prod\limits_{(i)} \varepsilon_i^{\mathcal{A}_{oi}}}{\prod\limits_{(\nu)} \prod\limits_{(i)} \Xi_\nu\, \varepsilon_i^*}$$

(2.3)

$$= \frac{\Xi^{A_{*o}}\, \varepsilon^{A_{o*}}}{\gamma(\Xi, \varepsilon)} \quad , \quad \text{say.}$$

From this we derive

$$p\left\{A_{*o}, A_{o*}\right\} = \begin{bmatrix} A_{*o} \\ A_{o*} \end{bmatrix} \frac{\Xi^{A_{*o}}\, \varepsilon^{A_{o*}}}{\gamma(\Xi, \varepsilon)}$$

where the bracket symbol - now with matrices inside - denotes the number of selection-vector matrices with the marginal total matrices given.

Furthermore with the notation

$$(2.5) \qquad \gamma_{A_{*o}}(\varepsilon) = \sum_{(A_{o*})} \begin{bmatrix} A_{*o} \\ A_{o*} \end{bmatrix} \varepsilon^{A_{*o}}$$

we get

$$(2.6) \qquad p\{A_{*o}\} = \frac{\Xi^{A_{*o}} \gamma_{A_{*o}}(\varepsilon)}{\gamma(\Xi, \varepsilon)}$$

and

$$(2.7) \qquad p\{A_{o*}|A_{*o}\} = \begin{bmatrix} A_{*o} \\ A_{o*} \end{bmatrix} \frac{\varepsilon^{A_{o*}}}{\gamma_{A_{*o}}(\varepsilon)}$$

as well as the symmetrical formula. Of course also

$$(2.8) \qquad p\{A|A_{*o}, A_{o*}\} = \frac{1}{\begin{bmatrix} A_{*o} \\ A_{o*} \end{bmatrix}} .$$

The analogy to (1.15) also holds but I need not elaborate that.

The generating functions are

$$(2.9) \qquad \begin{aligned} \gamma(\Xi, \varepsilon) &= \prod_{(\nu)} \prod_{(i)} \Big(\sum_{(\mu)} \xi_{\nu}^{(\mu)} \varepsilon_{i}^{(\mu)} \Big) \\ &= \sum_{(A_{*o})} \sum_{(A_{o*})} \begin{bmatrix} A_{*o} \\ A_{o*} \end{bmatrix} \Xi^{A_{*o}} \varepsilon^{A_{o*}} \end{aligned}$$

and

$$(2.10) \qquad \gamma_{A_{*o}}(\varepsilon) = \prod (\gamma_{\mathfrak{R}}(\varepsilon))^{c_{\mathfrak{R}}}$$

where $c_{\mathfrak{R}}$ denotes the number of row vectors in A_{*o} that equals $\mathfrak{R}$, i.e. the number of persons with a given total $\mathfrak{R}$ of selection vectors. The $\gamma_{\mathfrak{R}}(\varepsilon)$'s are generalizations of the elementary symmetrical functions and they may in analogy to (1.19) be defined by the expansion

$$\prod_{i=1}^{k}\left(\sum_{\mu=1}^{m}\varepsilon_i^{(\mu)}x^{(\mu)}\right)=\prod_{i=1}^{k}(\mathcal{E}_i\mathcal{X}^*)$$

(2.11)

$$=\sum_{(\mathcal{R})}\gamma_{\mathcal{R}}(\mathcal{E})\mathcal{X}^{\mathcal{R}},\ \mathcal{X}=(x^{(1)},\ldots,x^{(m)})$$

3. So far everything is analogous to $m=2$. However, when generalizing even further, a sort of symmetry principle emerges.

Let person no. ν be exposed to item no. i a number of times $l_{\nu i}$. Then we get similar distribution functions depending upon coefficients (A^{L}_{*o},A_{o*}) , generated by

$$\gamma_L(\Xi,\mathcal{E})=\prod_{(\nu)}\prod_{(i)}(\Xi_\nu\varepsilon_i)^{l_{\nu i}}= \qquad L$$ (3.1)

$$=\sum_{(A_{*o},A_{o*})}\left\{\begin{matrix}L\\A_{*o},A_{o*}\end{matrix}\right\}\Xi^{A_{*o}}\,\varepsilon^{A_{o*}}_{A} \ .$$

Now multiply this function by $\frac{\mathcal{Z}^L}{L!}$, $L!$ denoting the product $\prod\prod l_{\nu i}!$, and sum over all non-negative integers for the $l_{\nu i}$'s . Then we get

$$\sum_{(A_{*o},A_{o*},L)}\frac{1}{L!}\left\{\begin{matrix}L\\A_{*o},A_{o*}\end{matrix}\right\}\Xi^{A_{*o}}\,\varepsilon^{A_{o*}}\mathcal{Z}^L$$

(3.2)

$$=\sum_{(L)}\frac{\gamma_L(\Xi,\varepsilon)}{L!}\mathcal{Z}^L=\sum_{(L)}\frac{\prod_{(\nu)}\prod_{(i)}\left(\sum_{(\mu)}\xi_\nu^{(\mu)}\varepsilon_i^{(\mu)}\zeta_{\nu i}\right)^{l_{\nu i}}}{\prod_{(\nu)}\prod_{(i)}l_{\nu i}\,!}$$

$$=\exp\left(\sum_{(\nu,i,\mu)}\xi_\nu^{(\mu)}\varepsilon_i^{(\mu)}\zeta_{\nu i}\right)$$

in which the ξ , ε and ζ 's occur symmetrically. Therefore, by exchanging the role of the ζ's with that of the ξ's or the ε's we shall get the same expression and this fact implies that the coefficients

$$(3.3) \qquad \frac{1}{L!}\binom{L}{A_{*o},A_{o*}} = \frac{1}{A_{*o}!}\binom{A_{*o}}{L,A_{o*}} = \frac{1}{A_{o*}!}\binom{A_{o*}}{A_{o*},L}$$

are symmetrical in A_{*o}, A_{o*} and L .

To interpret it: In this context the numbers of specified answers given by each person, the numbers of specified answers given to each item and the numbers of trials applied to each combination of person and item are exchangeable. Psychologically this sounds peculiar, but I do not think there is more to it than a purely algebraical relation. Even so it seems remarkable that determining the coefficients in the developments of the three products

$$(3.4a) \qquad \prod_{v=1}^{n}\prod_{i=1}^{k}\left(\sum_{\mu=1}^{m} x_v^{(\mu)} y_i^{(\mu)}\right)^{c_{vi}} ,$$

$$(3.4b) \qquad \prod_{i=1}^{k}\prod_{\mu=1}^{m}\left(\sum_{v=1}^{n} x_v^{(\mu)} z_{vi}\right)^{b_i^{(\mu)}} \quad \text{and}$$

$$(3.4c) \qquad \prod_{v=1}^{n}\prod_{\mu=1}^{m}\left(\sum_{i=1}^{k} z_{vi}\, y_i^{(\mu)}\right)^{a_v^{(\mu)}}$$

are equivalent problems. I wonder whether it may be related to known duality principles in combinatorics. Anyhow, the symmetry (3.3) looks promising, but so far I have had little luck in exploiting it.

4. In conclusion I may add a few words about the perspective of my subject.

In the presentation of my problems I have referred to a psychological situation of a somewhat special kind, but the type of models described covers a much larger field.

To see that we may apparently specialize (2.1) in order to meet the demand that a model should not depend on too many parameters.

If we introduce logaritmic versions of the parameters

$$(4.1)\qquad \begin{gathered}\theta_\nu^{(\mu)} = \log \xi_\nu^{(\mu)},\ \sigma_i^{(\mu)} = \log \varepsilon_i^{(\mu)}\\ \Theta_\nu = (\theta_\nu^{(1)},\dots,\theta_\nu^{(m)}),\ \ \Sigma_i = (\sigma_i^{(1)},\dots,\sigma_i^{(m)})\end{gathered}$$

the model takes the form

$$(4.2)\qquad p\{x^{(\mu)}|\nu,i\} = \frac{e^{\theta_\nu^{(\mu)}+\sigma_i^{(\mu)}}}{\lambda(\Theta_\nu,\Sigma_i)}\quad ,$$

and now we may in actual cases ask whether it is possible to reduce the number, m , of parameters per person and per item (apart from the trivial reduction to $m-1$).

The simplest case would seem to be that each person as well as each item is fully characterized by a one-dimensional parameter, θ_ν and σ_i , respectively, and accordingly that the responses $x^{(\mu)}$ are measured in metrics which are independent of the situation:

$$(4.3)\qquad \theta_\nu^{(\mu)} = \theta_\nu\varphi^{(\mu)}+\alpha^{(\mu)},\ \sigma_i^{(\mu)} = \sigma_i\psi^{(\mu)}+\beta^{(\mu)}\quad .$$

The model is then reduced to

$$(4.4)\qquad p\left\{x^{(\mu)}\right\} = \frac{e^{\theta_\nu\varphi^{(\mu)}+\sigma_i\psi^{(\mu)}+\delta^{(\mu)}}}{\lambda(\theta_\nu,\sigma_i)}\ ,\ \delta^{(\mu)} = \alpha^{(\mu)}+\beta^{(\mu)}$$

which turns out to be a particular version of the discrete case of Darmois-Koopman's exponential distribution type.

Of course θ_ν and σ_i need not be scalars, they may be vectors, in which case the products just have to be interpreted as inner products.

In this form the model appears to be as flexible as to cover almost every known field of statistical analyses where static models are indicated, and it has actually been employed in biology, psychology, sociology, demography, economics, and linguistics.

Therefore, to reach at a mastering of an adequate testing apparatus is urgent, and therefore I shall be grateful for suggestions that may lead to further progress.