An “Estimation Bias” Shootout in the Wild West: CMLE, JMLE, MMLE, PMLE

Three riverboat gamblers, Bruce, George and Ben, are discussing tomorrow’s sharpshooting contest between Annie Oakley and Lillian Smith. Today’s *Deadwood Pioneer* newspaper contains reports of their previous contests in Table 1 and of their contests with Frank Butler in Table 2.

Table 1. Previous Contests between Annie and Lillian. (1 = Winner)		
Year	Annie	Lillian
1888	1	0
1887	0	1
1886	1	0
Shooter’s Score	2	1

Table 2. Contests between Annie or Lillian and Frank. (1 = Winner)		
Year	Annie	Lillian
1888	1	0
1887	0	1
1886	1	0
Shooter’s Score	2	1

“Tomorrow it’s Annie against Lillian. Here’s how to get the odds correct.” says Bruce. “Let’s use PMLE[1]. In each Table, Annie and Lillian were in the same situation three times, once for each row. Annie won twice. Lillian won once. The odds in both Tables are 2/1. Annie and Lillian are ln(2/1) = 0.69 logits apart.”

“We get those same odds of 2/1 from both Tables using CMLE[2,3].” agrees George.

“JMLE[4] and MMLE[5] estimate that the odds for both Tables are 4/1” says Ben. “To produce the correct odds of 2/1 for the direct pairwise comparison of Annie and Lillian in Table 1, we must adjust for JMLE estimation bias[6]. However, the odds for the indirect pairwise comparison of Annie and Lillian in Table 2 are 4/1. JMLE/MMLE are correct. There is no JMLE estimation bias for Table 2.”

“No! No!” objects George. “JMLE estimates are always biased[7], even though the bias reduces quickly for larger datasets[8]. Ben, you are way off target!”

“It’s you guys who can’t shoot straight” says Ben. “Let’s redraw Table 2 so that each row is a separate contest of sharpshooters. Here it is in Table 3 where all the participants are columns and there is one row for each pairwise contest, exactly like Table 1. Let’s take Bruce’s PMLE logic for Table 1 and apply it to Table 3. All the row scores are 1. In the upper three contests, Annie scores 2 and Frank scores 1. The odds are 2/1 for Annie against Frank. In the lower three contests, Frank scores 2 and Lillian scores 1. The odds are 2/1 for Frank against Lillian. Combining these, the odds for Annie against Lillian are (2/1) * (2/1) = 4/1. They are ln(4/1) = 1.39 logits apart, exactly as JMLE tell us!”

Table 3. Contests between Annie or Lillian and Frank (redrawn). (1 = Winner)			
Year	Annie	Frank	Lillian
1888	1	0	
1887	0	1	
1886	1	0	
1888		1	0
1887		0	1
1886		1	0
Shooter’s Score	2 of 3	3 of 6	1 of 3

“That is exciting!” exclaims Bruce. “We can extend Table 3 to much larger competitive situations such as Basketball[9] and Tennis using PMLE or bias-adjusted JMLE[10].”

“Ben, why didn’t you explain this to me years ago?” says George. “In Tables 1 and 3, the paired comparisons are direct. In Table 2 the comparisons are indirect. Table 2 is an abbreviated version of Table 3. When we treat the comparisons in Table 2 as direct, we distort the meaning of the data, resulting in biased estimates.”

“Exactly!” says Ben, “When a dataset is directly pairwise, as in Tables 1 and 3, CMLE/PMLE estimates are accurate and unbiased. We must bias-adjust JMLE estimates. For a

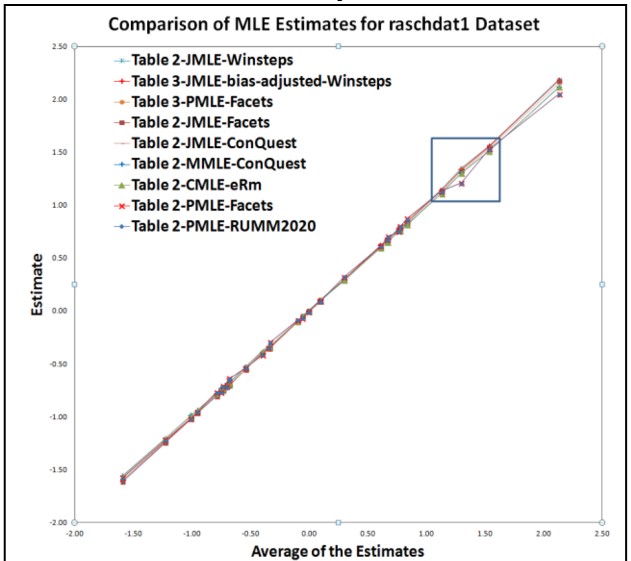

Figure 1. MLE estimates for raschdat1

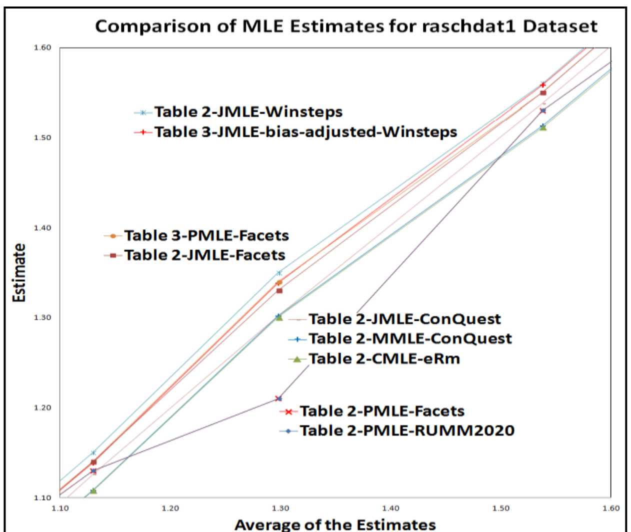

Figure 2. MLE estimates for raschdat1 (Excerpt).

dataset of indirect comparisons like Table 2, JMLE estimates are unbiased. CMLE/PMLE estimates for any dataset are biased if reformatting that dataset to be directly pairwise produces different CMLE/PMLE estimates."

Dear Reader: Would you like more evidence? raschdat1.rda is a dichotomous dataset of 30 items and 100 persons distributed in the *eRm* package. In www.rasch.org/rmt/a/shootout.zip, there are conventional Table 2 (30x100) versions of raschdat1 for CMLE, JMLE, MMLE, and PMLE, also Table 3 pairwise (130x3000) versions for JMLE and PMLE, together with their estimates (see Figure 1) and the Excel worksheet of the Figures. In Figure 2, Table 3 curves track with Table 2 JMLE curves, confirming Ben's claim that JMLE is unbiased for conventional datasets.

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Endnotes:

[1] PMLE, Pairwise Maximum Likelihood Estimation, in *RUMM2030* and *Facets.*

[2] CMLE, Conditional Maximum Likelihood Estimation, in *eRm.*

[3] *CMLE for Tables 1 and 2:* All the rows are scored 1 on the 2 items, so we only need the probabilities for a row score of 1. Let's call P(10) the probability that Annie wins and Lillian loses, then P(01) is the opposite. For each row the total probability for a score of 1 is P(10) + P(01). In each Table, Annie scored 2 in 3 attempts, so 2 = 3 * P(10) / (P(10) + P(01)). Lillian scored 1 in 3 attempts, so 1 = 3 * P(01) / (P(10) + P(01)). Now, divide those two equations, then the odds are P(10)/P(01) = 2/1. ln(2/1) = 0.69 logits.

[4] JMLE, Joint Maximum Likelihood Estimation, in *ConQuest, Facets* and *Winsteps.* The JMLE estimates are the ones for which the observed marginal score equals the expected marginal score for each row and column.

[5] MMLE, Marginal Maximum Likelihood Estimation, in *ConQuest.* MMLE estimates are the ones for which the observed marginal score for each column equals the expected marginal score, and the row parameters are modeled to have a normal distribution.

[6] Using *Winsteps, JMLE* pairwise estimation bias is adjusted by *Paired=Yes*

[7] Andersen E.B. (1970) Asymptotic properties of conditional maximum likelihood estimators. Journal of the Royal Statistical Society B 32, 283–301

[8] Wright, B.D. (1988) The efficacy of unconditional maximum likelihood bias correction: Comment on Jansen, Van den Wollenberg, and Wierda. Applied Psychological Measurement, 12, 315-318.

[9] Linacre J.M. (2001) Paired comparisons for measuring team performance. RMT, 15:1, 812.
www.rasch.org/rmt/rmt151w.htm

[10] Linacre J.M. (1997) Paired comparisons with standard Rasch software. RMT, 11:3, 584-5.
www.rasch.org/rmt/rmt113o.htm