CMLE – a Problem, its Solution and a Useful Approximation

CMLE, Conditional Maximum Likelihood Estimation, produces statistically consistent item estimates for Rasch data, so that, as the person sample size increases, the CMLE item estimates converge to their true values. However there is a problem. The person theta estimates are not congruent with the item estimates. This is because CMLE, as usually implemented, only produces item estimates. These item estimates are then used as anchor values in AMLE, Anchored Maximum Likelihood Estimation, to produce theta estimates. The solution is to use CMLE for both the persons and the items. Both of these CMLE estimates can be approximated from JMLE results. Here are the details.

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1.	*Concept*	*Details*
2.	**Situation 1:**	**The dichotomous data are symmetric**
3.	Dataset 1: 4 items, E1, E2, E3, E4 4 persons, P1, P2, P3, P4	<table><tr><td></td><td>E1</td><td>E2</td><td>E3</td><td>E4</td><td>Total</td></tr><tr><td>P1</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td></tr><tr><td>P2</td><td>0</td><td>1</td><td>1</td><td>0</td><td>2</td></tr><tr><td>P3</td><td>0</td><td>1</td><td>1</td><td>1</td><td>3</td></tr><tr><td>P4</td><td>0</td><td>0</td><td>1</td><td>1</td><td>2</td></tr><tr><td>Total</td><td>1</td><td>2</td><td>3</td><td>2</td><td></td></tr></table>
4.	CMLE item estimates produced by R Statistics package, eRm	CMLE Item Easiness E1 = -0.955, E2 = 0.000, E3 = 0.955, E4 = 0.000 Item Difficulty = - Item Easiness
5.	Internally in eRm	Anchor the item estimates at their CMLE values, then
6.	AMLE person estimates produced by eRm	AMLE Ability Theta P1 = -1.209, P2 = 0.000, P3 = 1.209, P4 = 0.000
7.	Logit ranges	CMLE Item range: 1.910 AMLE Person range: 2.418 Different!
8.	**Problem: Biased person estimates**	The data are symmetric The item and person estimates are not symmetric If CMLE estimates are unbiased, then AMLE estimates must be biased
9.	Solution	CMLE of items Transpose the data matrix CMLE of persons
10.	CMLE person estimates produced by eRm	CMLE Ability Theta P1 = -0.955, P2 = 0.000, P3 = 0.955, P4 = 0.000

<table>
<tr><td>11.</td><td>Logit ranges</td><td>CMLE Item range: 1.910
CMLE Person range: 1.910
The same!</td></tr>
<tr><td>12.</td><td>Solution 1:</td><td>1. AMLE of person estimates is biased
2. CMLE person estimation solves this problem</td></tr>
<tr><td>13.</td><td>Situation 2:</td><td>An asymmetric dichotomous dataset</td></tr>
<tr><td>14.</td><td>Dataset 2:
4 items, E1, E2, E3, E4
5 persons, P1, P2, P3, P4, P5</td><td>
<table>
<tr><td></td><td>E1</td><td>E2</td><td>E3</td><td>E4</td><td>Total</td></tr>
<tr><td>P1</td><td>1</td><td>0</td><td>0</td><td>0</td><td>1</td></tr>
<tr><td>P2</td><td>0</td><td>1</td><td>0</td><td>1</td><td>2</td></tr>
<tr><td>P3</td><td>0</td><td>0</td><td>1</td><td>1</td><td>2</td></tr>
<tr><td>P4</td><td>0</td><td>1</td><td>1</td><td>1</td><td>3</td></tr>
<tr><td>P5</td><td>0</td><td>1</td><td>1</td><td>1</td><td>3</td></tr>
<tr><td>Total</td><td>1</td><td>3</td><td>3</td><td>4</td><td></td></tr>
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</td></tr>
<tr><td>15.</td><td>CMLE item estimates produced by eRm</td><td>CMLE Item Easiness
E1 = -1.385, E2 = 0.160, E3 = 0.160, E4 = 1.065</td></tr>
<tr><td>16.</td><td>Since AMLE is biased, here are CMLE person estimates produced by eRm</td><td>CMLE Ability Theta (transposed matrix)
P1 = -1.307, P2 = -0.284, P3 = -0.284, P4 = 0.937, P5 = 0.937</td></tr>
<tr><td>17.</td><td>Aligning CMLE person and item estimates</td><td>The means of both the item and person estimates are set to 0.0, so the person estimates need adjusting for overall performance of persons relative to items. We use the probability matrices to do this.</td></tr>
<tr><td>18.</td><td>Item CMLE: Here is the matrix of expected probabilities at the end of item estimation, before person estimation. This matrix is almost symmetric. (Not produced by eRm at time of writing.)</td><td>
<table>
<tr><td></td><td>E1</td><td>E2</td><td>E3</td><td>E4</td><td>Total</td></tr>
<tr><td>P1</td><td>0.046</td><td>0.213</td><td>0.213</td><td>0.528</td><td>1.000</td></tr>
<tr><td>P2</td><td>0.138</td><td>0.534</td><td>0.534</td><td>0.793</td><td>2.000</td></tr>
<tr><td>P3</td><td>0.138</td><td>0.534</td><td>0.534</td><td>0.793</td><td>2.000</td></tr>
<tr><td>P4</td><td>0.339</td><td>0.859</td><td>0.859</td><td>0.943</td><td>3.000</td></tr>
<tr><td>P5</td><td>0.339</td><td>0.859</td><td>0.859</td><td>0.943</td><td>3.000</td></tr>
<tr><td>Total</td><td>1.000</td><td>3.000</td><td>3.000</td><td>4.000</td><td></td></tr>
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<tr><td>19.</td><td>Person CMLE: Here is the matrix of expected probabilities at the end of person estimation. This matrix is almost symmetric, but is not the same as the item probabilities matrix</td><td>
<table>
<tr><td></td><td>E1</td><td>E2</td><td>E3</td><td>E4</td><td>Total</td></tr>
<tr><td>P1</td><td>0.039</td><td>0.239</td><td>0.239</td><td>0.482</td><td>1.000</td></tr>
<tr><td>P2</td><td>0.109</td><td>0.538</td><td>0.538</td><td>0.814</td><td>2.000</td></tr>
<tr><td>P3</td><td>0.109</td><td>0.538</td><td>0.538</td><td>0.814</td><td>2.000</td></tr>
<tr><td>P4</td><td>0.371</td><td>0.842</td><td>0.842</td><td>0.945</td><td>3.000</td></tr>
<tr><td>P5</td><td>0.371</td><td>0.842</td><td>0.842</td><td>0.945</td><td>3.000</td></tr>
<tr><td>Total</td><td>1.000</td><td>3.000</td><td>3.000</td><td>4.000</td><td></td></tr>
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<tr><td>20.</td><td>Solution 2: Alignment procedure for CMLE person and item estimates using the probability matrices.</td><td>The means of both the CMLE item and CMLE person estimates are set to 0.0, so the person distribution needs aligning to the item distribution.
Step 1) Look for a cell with probability X near 0.5.
Comparing these two probability matrices, we see that for P1/E4 the values bracket .50, so let's average them: X = (0.528+0.482)/2 = 0.505
Step 2) Identify the CMLE item easiness and CMLE person ability estimates for the cell. Example: E4 = 1.065, P1 = -1.307
Step 3) Add to the CMLE person estimates:
Adjustment = ln(X/(1-X)) - CMLE Item Easiness - CMLE Person Ability
Example: = ln(0.505 / 0.495) - 1.065 - -1.307 = 0.262 logit adjustment
Aligned CMLE Ability Theta
P1 = -1.045, P2 = -0.022, P3 = -0.022, P4 = 1.199, P5 = 1.199</td></tr>
<tr><td>21.</td><td>A useful approximation</td><td>CMLE estimates from CMLE or JMLE probabilities</td></tr>
<tr><td>22.</td><td>JMLE: Here is the matrix of expected probabilities for Dataset 2 at the end of item and person estimation. This matrix is symmetric.</td><td>
<table>
<tr><td></td><td>E1</td><td>E2</td><td>E3</td><td>E4</td><td>Total</td></tr>
<tr><td>P1</td><td>0.032</td><td>0.232</td><td>0.232</td><td>0.505</td><td>1.000</td></tr>
<tr><td>P2</td><td>0.115</td><td>0.542</td><td>0.542</td><td>0.800</td><td>2.000</td></tr>
<tr><td>P3</td><td>0.115</td><td>0.542</td><td>0.542</td><td>0.800</td><td>2.000</td></tr>
<tr><td>P4</td><td>0.369</td><td>0.842</td><td>0.842</td><td>0.947</td><td>3.000</td></tr>
<tr><td>P5</td><td>0.369</td><td>0.842</td><td>0.842</td><td>0.947</td><td>3.000</td></tr>
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<tr><td>23.</td><td>Coincidences between cell probabilities: CMLE items, CMLE persons, JMLE</td><td>1) Marginal item (column) and person (row) totals are the same for all three probability matrices. Though JMLE estimates are more dispersed than CMLE estimates, their totals are the same.
2) Probabilities in the cells of all three matrices are functionally the same. Conclusions based on cell probabilities, such as standard errors and mean-square fit statistics, are effectively the same for JMLE and CMLE.
JMLE cell probabilities approximate CMLE cell probabilities.</td></tr>
<tr><td>24.</td><td>The CMLE theory</td><td>For a person score of 1, for any pair of items, here items 1 and 2,
Exp(E1) / Exp(E2) = Probability (E1)/Probability(E2)
Example using CMLE item probabilities, person P1, score 1,
Exp(E1) / Exp(E2) = Exp(-1.385)/Exp(0.160) = 0.213
Probability (E1)/Probability(E2) =0.046/0.213 = 0.216
These values are effectively equal. CMLE theory confirmed.</td></tr>
<tr><td>25.</td><td>Approximate CMLE estimates from CMLE/JMLE probabilities</td><td>Since all the probability matrices are similar, approximate CMLE item or CMLE person estimates can be obtained from CMLE item or CMLE person or JMLE cell probabilities for a person or item with a score of 1.</td></tr>
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26.	Approximate CMLE item estimates from the CMLE person or JMLE probability matrices	Step 1) Using a CMLE person or JMLE probability matrix, identify or generate a person with a score of 1. Example: P1 Step 2) Compute the probabilities (expected values) for every item. Example from the JMLE matrix: 0.039 0.232 0.232 0.505 Step 3) Identify a probability, Pmiddle, in the middle of the range. Assign its item an estimate of 0 logits with Exp (0) = 1. Example: Pmiddle = 0.232, E2 = 0.000 logits Step 4) The CMLE estimates for all the other items are: Target Item estimate = ln (target item probability / Pmiddle) Example: E1, E2, E3, E4 = -1.783, 0.000, 0.000, 0.778 Step 5) Subtract the mean of all the item estimates from each of the item estimates. These are now the approximate CMLE item estimates. Example: mean = -0.251 E1 = -1.532, E2 = 0.251, E3 = 0.251, E4 = 1.029
27.	Three sets of "CMLE" item estimates:	Exact: CMLE Item Easiness directly or from CMLE item probabilities E1 = -1.385, E2 = 0.160, E3 = 0.160, E4 = 1.065 Approximate: CMLE Item Easiness from CMLE person probabilities E1 = -1.535, E2 = 0.278, E3 = 0.278, E4 = 0.979 Approximate: CMLE Item Easiness from JMLE probabilities E1 = -1.532, E2 = 0.251, E3 = 0.251, E4 = 1.029 Note: in this example, item estimates from JMLE probabilities are closer to exact item CMLE than item estimates from CMLE person probabilities.
28.	Approximate CMLE person estimates	Same estimation procedure as above using the CMLE item or JMLE probabilities for an item with a score of 1.
29.	Three sets of "CMLE" person estimates (unaligned):	Exact: CMLE Ability Theta directly or from CMLE person probabilities P1 = -1.307, P2 = -0.284, P3 = -0.284, P4 = 0.937, P5 = 0.937 Approximate: CMLE Ability Theta from CMLE item probabilities P1 = -1.238, P2 = -0.140, P3 = -0.140, P4 = 0.759, P5 = 0.759 Approximate: CMLE Ability Theta from JMLE probabilities P1 = -1.331, P2 = -0.250, P3 = -0.250, P4 = 0.916, P5 = 0.916 Note: in this example, person estimates from JMLE probabilities are closer to exact person CMLE than person estimates from CMLE item probabilities.
30.	Aligning CMLE person and CMLE item estimates (exact or approximate)	Same alignment procedure as above. Notice that for P1/E4 the JMLE probability is 0.505, the average value we obtained above.

31.	Advantages of using the JMLE probability matrix to obtain approximate CMLE estimates	1) JMLE computation is faster for large matrices 2) JMLE easily accommodates missing data 3) JMLE probability matrix and JMLE-based CMLE estimates are symmetric for symmetric data 5) Using zero-weighted dummy items and persons, JMLE generates probability matrices which include all possible person and item scores for CMLE-compatible item and person score-to-measure tables.